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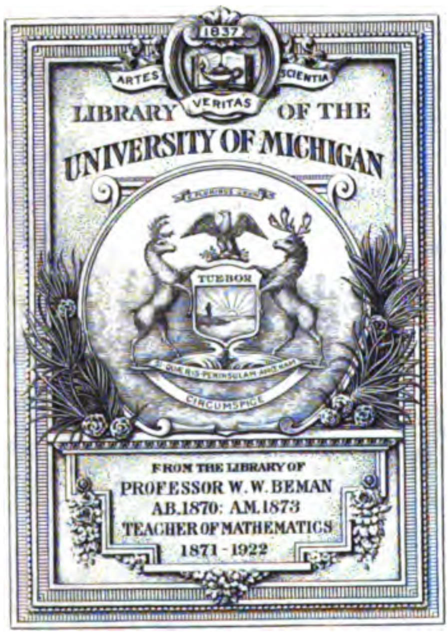
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NOVI SYSTEMATIS

PERMUTATIONUM COMBINATIONUM AC VARIATIONUM

PRIMAE LINEAE

ET

LOGISTICAE SERIERUM

FORMULIS ANALYTICO-COMBINATORIIS

PER TABULAS EXHIBENDAE

CONSPECTUS ET SPECIMINA

AUCTORE

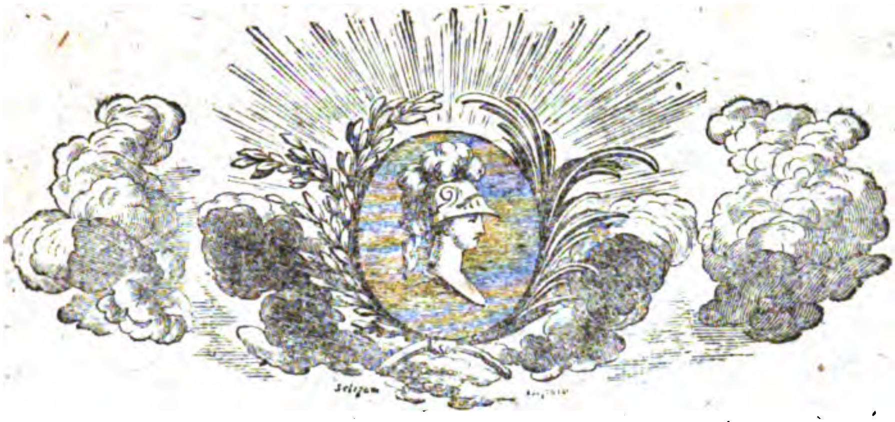
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PRAEFATIO

Artis combinatoriae primas lineas et Logiſticae Serierum conſpectum in Proluſione academica nuper propoſui; quam denuo nunc recudendam curavi, adiecta, auctarii loco, expoſitione paulo ampliori, additis etiam Tabularum ſpeciminibus; unde, quae ſit futura res, et quomodo in opere ipſo, cuius ſpem feci, maiore tractabitur, intelligi plane poſſit ac penitus perſpici.

Ars Combinatoria Permutationum, Complicationum ac Variationum a rebus datis exponit Formas ſingulas et Numerum formarum, ſive res admittantur repetitae ſive non admittantur, ſive ſint unius generis omnes ſive etiam plurium. Hinc pendent Elementa, quae dixi, combinatoria, quae, literarum ope aut numerornm, quomodo colligi in Claſſes, et quo ordine diſponi debeant, ut progrediantur eo modo, quo numeri creſcunt, ex ſingulis figuris vel ſimpliciter, vel quoad ſummam propoſitam, compoſiti, ex Tabulis optime

* a appa-

427030

apparet hoc libello propoſitis combinatoriis, paucis illis quidem ex pluribus, quas afferemus ſuo loco, ſed maxime univerſalibus et latiſſime patentibus.

Uſum rei combinatoriae in omni ſcientiarum et artium genere plane ſingularem, nemo eſt qui ignorat. In his Analyſis haud dubie primum locum occupat, propter ſummam neceſſitudinem et interiorem aliquam coniunctionem, quae ei cum Arte combinatoria intercedit. Pone res propoſitas quotcunque; oſtendet Ars combinatoria harum quomodo coniunctiones poſſibiles ſingulae ac transpoſitiones abſolvi poſſint, numerusque coniunctionum et transpoſitionum reperiri: Pone problematis quantitates datas quotcunque; oſtendet Analyſis quantitates quaeſitae cum Coefficientibus numericis quomodo inde pendeant, ex datarum tandem vel coniunctionibus vel permutationibus definiendae. Ars igitur combinatoria rerum coniungendarum transponendarum formas ſingulas ex ordine proponit ſeorſim: Analyſis easdem, adhibitis operationibus et ſignis arithmeticis, collectas, quibus opus eſt, diſponit coniunctim. Analyſis oſtendit, quae ſunt agenda: Ars combinatoria, quomodo ſunt agenda. Refert itaque et plurimum prodeſt ad Analyſin, Operationes noſſe combinatorias; Tabulis, ut ego feci, concludere, per elementa, quibus abſolvuntur, combinatoria, optimum haud dubie videatur, certe ad uſum maxime accommodatum. Sic Tabulae Arti combinatoriae propriae, eaedem ſunt Analyſi ſubſidiariae, et aeque late patent in Analyſi, quam Tabulae ſinuum ac logarithmorum in Trigonometria et Geometria practica. Differunt ab his Tabulae analyticae, formulas ipſas analytico-combinatorias ſive canones complectentes, quibus continetur Logiſtica Serierum.

De

De diſcrimine, quod obſervatur, in Problematum circa Series Analyſi communi et combinatoria, ſic habeto. Analyſis communis Operationes algebraicas aut tranſcendentes in ipſis quantitatibus datis inſtituendas, praeſcribit ſic, ut peractis demum, more conſueto, his operationibus appareat, quae tandem quantitates et quomodo componantur inter ſe aut permiſceantur, h. e. combinentur aut permutentur: Analyſis contra combinatoria, combinationes ipſas aut permutationes (a Multiplicationibus, Diviſionibus, Subſtitutionibus, ad Poteſtates Evectionibus, Radicum Extractionibus, aut aliis quibuscunque Operationibus oriundas) ſtatim proponit, ex partibus integrantibus, eo loco, quo manere, eaque forma, quam ſervare debent. Analyſis communis ad ſubſtitutiones haud raro ac reductiones operoſiores taediique plenas deducit: Analyſis combinatoria easdem vel plane tollit, vel longe tamen reddit faciliores. Illa terminos extra ordinem exhibere ſaepenumero non poteſt: haec poteſt, in quantitatibus adeo ſimpliciſſimis. Illa Legem progreſſus formula idonea exprimere interdum non valet: haec proponit ſumma cum perſpicuitate. Adde opportunitatem, quae redundat ex ſignorum, quas exhibui, relationibus mutuis; adde, quod ipſae formae ſive ſpecies combinatoriae, vel ſingulae, vel collectae in Claſſes, Calculi analytici vere exhibeant Elementa, quae per Tabulas diſpoſita, adeoque iamiam parata, ex his ipſis Tabulis commode poſſunt, ubi opus eſt, ad uſum depromi; adde, quod ipſe Computus numerorum, adhibitis eorum figuris ſingulis, tanquam elementis, inde iuvetur: et apparebit, opinor, ſatis, qui fructus ex applicatione Artis combinatoriae ad Analyſin percipi poſſint. Nempe tollitur hoc modo calculorum fere immenſorum operoſum ac permoleſtum negotium, formulis exhibendis elementaribus,

et progreſſus Legem, harmonia quadam ſua, aut ſymmetria, adeo manifeſte exponentibus, ut, qua tranſitus pateat a ſimplicioribus ad magis compoſita, obſcurum eſſe nullo modo poſſit. Scilicet ordinis et concinnitatis una omnium ſervantiſſima eſt Ars combinatoria, et campum aperit obſervationibus ac meditationibus longe lateque vaſtiſſimum.

Diſpiciendum erat itaque ante omnia, Operationes notiſſimae illae, cum ipſis Seriebus inſtituendae: Serierum in Series Multiplicatio, Serierum per Series Diviſio, Serierum ad Poteſtates Evectio et Radicum ex Seriebus Extractio, Serierum in Series Subſtitutio; hae, inquam, operationes, et formulae, quae inde fluunt, algebraicae aut tranſcendentes, quomodo per operationes ac formulas combinatorias, Elementorum combinatoriorum ope, exhiberi poſſent. Haec ſcilicet ſunt Calculi analytico-combinatorii fundamenta, quae Tabulis quatuor prioribus proponere conſtitui. Ab his reliquae Tabulae pendent, quippe formulas complectentes, ab operationibus, quas modo memoravi, oriundas. Tabulae primae et ſecundae, Serierum ſcilicet Multiplicationis et Diviſionis exhibita ſunt ad calcem libri ſpecimina.

Artis combinatoriae, nova methodo tractandae et ad Analyſin applicandae, principia complura iam extant in libro, quem de Infinitinomii Dignitatibus, duobus abhinc annis, edidi; ad quem propterea Lectores huius libelli ſaepenumero ablegavi. Nempe impraeſentiarum Totius volui aliquam tantum adumbrationem, non rem ipſam exponere; in elaborando autem atque adornando ipſo Combinationum ac Permutationum novo Syſtemate, operam dabo, ut principia

poſita

poſita ſuo loco ac ordine appareant certa atque evidentiſſima; ut omnia ad propoſitiones ſimpliciſſimas revocentur; omnia iungantur eo ordine, qui rebus quaerendis et determinandis maxime ſit conveniens; ut Complicationum ac Permutationum, a rebus pluribus oriundarum, exponantur ipſae etiam Formae, non ſolum Numerus formarum; ut nihil denique, quoad eius fieri poteſt, praetermittatur eorum, quibus poſſit Analyſin iuvare Ars combinatoria.

De Semiotica ſive Characteriſtica, cuius in utraque Arte, combinatoria et analytica, inprimis eſt habenda ratio, pluribus hic dicere nihil attinet, cum de ea diligenter iamiam in ipſo libello expoſitum ſit. Adhibui autem notationis modum eiusmodi, qui notiones, quamtumvis etiam compoſitas, facillime animo repraeſentet ac veluti contuendas praebeat, qui Terminorum ſingulorum progreſſum (pro ſignorum, ſi quae ſunt, viciſſitudinibus, factorum porro aut diviſorum, Claſſium combinatoriarum aut Specierum ſingularum, Coefficientium denique binomialium, polynomialium et compoſitorum ex his, ordine ac conſecutione) manifeſte ante oculos ponat, ſic, ut ex formulis analytico-combinatoriis, termini ſive terminorum coefficientes quaeſiti quicunque, ex ordine et extra ordinem, a praecedentibus terminis vel coefficientibus independenter, in quantitatibus ſtatim ſimpliciſſimis, per exponentes aut coefficientes datos particulares definitis, absque ullo fere negotio poſſint derivari, problematum etiam ſolutiones, et terminorum, quibus abſolvuntur, progreſſus Leges exhiberi, quae alia via aliaque ratione interdum fruſtra tentantur.

Species combinatoriae, quae formulas analytico-combinatorias ingrediuntur, facile poſſunt ſingulae, in numeris ſive literis, ſecundum regulas traditas, exhiberi, et proinde nullo modo pendent a Tabulis neceſſario. Sed expedit, ſi ſpecies combinatoriae, quibus opus eſt, non debent pro ſingulis caſibus ipſae formari, ſed ſtatim poſſunt, cum ipſarum comitibus, ſive coefficientibus numericis h. e. transpoſitionum numeris, ex Tabula aliqua, ad hunc uſum parata, depromi. Huiusmodi Tabulas, tam combinatorias quam analyticas, in ipſo opere complures excudendas curabimus et latiſſime patentes. Quae exhibentur hoc libello, ſpeciminis tantum loco poſitae ſunt, ut, quid ſit res et quem habeat uſum, appareat.

Quae Analyſi poſſint ac debeant ex Combinatoria Arte commoda et emolumenta accedere, longe, quam dici poteſt, maiora, primus vidit LEIBNITIUS, Vir Summus. Multis hoc et indubiis conſtat argumentis, de quibus pluribus dixi in praefatione ad libellum de Infinitinomiorum Dignitatibus. In his commemoratur etiam Problematis de Serierum Regreſſu maxime implicati ſolutio; qua, quid Ars combinatoria in Analyſi, quid numeri efficere valeant, pro terminis ipſis, ſive terminorum coefficientibus poſiti, oſtendere allaboravit. Sed tamen, quod mirum fortaſſe videbitur et vix credibile, res caruit ſucceſſu, quantumvis eam LEIBNITIUS verbis multis et efficaciſſimis depraedicaverit, laudibus tantum non in coelum extulerit. Scilicet rem ipſam quidem viderat ac penitus perſpexerat Vir Magnus, ſed, quo minus poſſet eius pulchritudinem et utilitatem aliis demonſtrare, impediebat, quod problemata potiora combinatoria, quibus v. c. Complexiones aut Variationes numeri propoſiti poſtulantur,

non-

nondum in poteſtate habebat, notatione etiam, bona alias et commoda, minus dextre utebatur; ponendo ſcilicet, non characteres idoneos, quibus, quid fieri debeat numerorum ſubſidio, imperetur, ſed numeros ipſos fictitios aut ſuppoſitios in formulam, ſic ut, multitudine terminorum hoc modo provenientium, nulla omnino progreſſus Lege propoſita, formulam potius obrueret, quam nova deſignatione illuſtraret. Quid? quod Ipſi IOANNI BERNOULLIO res nimis abſtracta et abſtruſa viſa eſt. Atque hinc, facto primo periculo, quod ex voto non ſucceſſerat, cum ſubſiſteret LEIBNITIUS, foetumque adeo, quantumvis egregium, in partu, ut aiunt, deſtitueret, evenit, quod rebus humanis ſolet: res ignota deſpicatui primum habita, poſt eſt adducta in oblivionem, et regionibus analyticis hac parte gravis rurſus incubuit nox.

Poſt LEIBNITIUM, qui mihi quidem notus eſſet, nemo extitit, qui de Analyſi ex Combinatoria arte perficienda ſerio atque inſtituta opera cogitaſſet. CRAMERUM quidem in eliminationibus incognitarum, BOSCOVICIUM autem et CASTILLIONEUM in polynomiorum potentiis quaerendis, numeris etiam earumque combinationibus ac permutationibus uſos reperi: ſed res tamen ſemper intra nimis arctos cancellos ſubſtitit, tantum abfuit, ut problemata ipſa combinatoria vera ac genuina proponerent, propoſitaque ad Analyſin, methodi generalis ope, applicarent. Et quod magis etiam mireris, neque BOSCOVICIUS, neque CASTILLIONEUS, neque etiam ipſe CRAMERUS (qui longius reliquis progreſſus videtur, LEIBNITII etiam itemque IO. et IAC. BERNOULLIORUM tempora propius attigit) eo loco, ubi problematum propoſitorum ſolutionibus combinationes adhibent aut

permutationes, earumque uſum egregie collaudant et commendant, LEIBNITII mentionem ullam faciunt; ut credibile ſit, eos per ſe et forte fortuna (ut mihi quoque accidit, cum de Coefficientis Infinitinomialis EULERO-KAESTNERIANI ulteriori Analyſi cogitarem) incidiſſe in huiusmodi operationes; nempe, ut LEIBNITIUS ait, nihil eſt facilius, nihil naturae rei conſentaneum magis. Et profecto, quid ſit res et quas habeat opportunitates, nemo facile ex Leibnitiano exemplo, ſpeciminis loco propoſito, intellexerit, nemo inde methodum aliquam generalem, qua Artis combinatoriae praecepta Analyſi accommodari poſſint, Analyſis ex Combinatoria perfici, facile exſculpſerit.

Ceterum quae expectaverit emolumenta ab Arte Characteriſtica et Combinatoria vera et perfecta LEIBNITIUS, et quae inde ad Analyſin redundare poſſint commoda, praeter loca, quae in Praefatione ad libellum de Infinitinomiorum Dignitatibus, partim indicavi, partim etiam excerpta dedi, intelligi optime poterit ex his, quae ſequuntur:

Algebra, quam tanti facimus merito, generalis alius artificii non niſi pars eſt. Id tamen praeſtat, Errare ne poſſimus quidem, ſi velimus. Et, ut veritas quaſi picta, velut Machinae ope in charta expreſſa, deprehendatur. Ego vero agnoſco, quidquid in genere probat Algebra, non niſi ſuperioris ſcientiae *beneficium eſſe, quam nunc COMBINATORIAM CHARACTERISTICAM appellare ſoleo: longe diverſam ab illa, quae auditis his vocabulis, ſtatim alicui in mentem venire poſſet. Huius mirabilem vim ac poteſtatem, praeceptis aliquando et ſpeciminibus me explicaturum ſpero, ſi ſanitas atque otium fuerit.*

Non

Non poſſum paucis verbis rei naturam complecti. Illud tamen dicere auſim, Nihil facile ad humanae mentis perfectionem efficacius concipi poſſe. Ac, recepta hac philoſophandi ratione, fore tempus, et mox fore, quo de Deo ac Mente non minus certa, quam de figuris numerisque habeamus: quo Machinarum inventio non difficilior, quam conſtructio problematum Geometricorum. Exhauſtisque his ſtudiis (niſi quod ſemper infinitorum theorematum elegantiſſimae ſupererunt harmoniae; indies tunc obſervandae magis, quam eruendae) ad ſolam hominis redibunt naturae indagationem; quae nunquam in [fortunae] *poteſtate futura eſt. Nam, in experimentis ingenii et induſtriae fortuna miſcetur* ([a]).

Conſilium etiam habeo TABULARUM ANALYTICARUM, quae non minoris futurae eſſent uſus in Analyſi, quam Tabulae Sinuum in Geometria practica; imo, arbitror, qui paulum in iis calculandis verſatus ſit, eum progreſſiones reperturum in infinitum, quarum ope magna Tabulae pars ſine labore continuari poſſit. Nihil eſt, quod norim in tota Analyſi momenti maioris. Nam in his Tabulis pleraque problemata ſtatim ſoluta haberentur, aut levi opera poſſent inde deduci.

Pendet negotium ex re longe maiore; ARTE ſcilicet COMBINATORIA generali *ac* vera. *Cuius vim ac poteſtatem neſcio an quisquam hactenus ſit conſequutus. Ea vero nihil differt ab* Analyſi *illa* ſuprema, *ad cuius intima, quantum iudicare poſſum, CARTESIUS non pervenit. Eſt enim ad eam conſtituendam opus Alphabeto cogitationum humanarum, et ad inventionem eius Alphabeti opus eſt Analyſi axiomatum. Sed non miror, iſta nemini ſatis conſiderata; Quia plerumque facilia negligimus; et multa, quae clara videntur aſſumimus. Quod quamdiu faciemus, nunquam ad illud perveniemus, quod mihi videtur in rebus intellectualibus ſummum; nec genus Calculi etiam non-Mathematicis accommodati, obtinebimus* ([b]).

Intima

(*a*) LEIBNIT. Opp. T. III. p. 34, 35.

(*b*) LEIBNIT. Opp. T. III. p. 54.

Intima Variationum doctrina ſola prope per omne infinitum obſequentem ducit animum; et harmoniam mundi et intimas conſtructiones rerum, ſeriemque formarum una complectitur. Cuius incredibilis utilitas perfecta demum philoſophia, aut prope perfecta, recte aeſtimabitur. — Complicationibus, non ſolum infinitis novis theorematis locupletari geometria poteſt, nova enim Complicatio novam figuram compoſitam efficit, cuius iam contemplando proprietates, nova theoremata, novas demonſtrationes fabricamus; ſed et, (ſiquidem verum eſt, grandia ex parvis, ſive haec atomos, ſive moleculas voces, componi) unica iſta via eſt, in arcana naturae penetrandi. Quando eo quisque perfectius rem cognoſcere dicitur, quo magis rei partes et partium partes, earumque figuras poſitusque percepit. Haec figurarum ratio primum abſtracte in geometria ac ſtereometria perveſtiganda: inde ubi ad hiſtoriam naturalem exiſtentiamque, ſeu id, quod revera invenitur in corporibus, acceſſeris, patebit Phyſicae porta ingens; et elementorum facies, et qualitatum origo, et mixtura, et mixturae origo, et mixtura mixturarum, et quidquid hactenus in natura ſtupebamus (c).

(c) LEIBNIT. Opp. T. II. p. 360, 361.

§. I.

Ars combinatoria cum in omni Analyſi, tum vero potiſſimum Serierum tractatione inſignem uſum praeſtat.

Formulae analyticae, quibus problematum efferri ſolent ſolutiones, terminis ubi conſtant per datas ipſas quantitates ſimpliciſſime expreſſis, apteque et concinne diſpoſitis, optime oſtendunt, quae quantitatum ſit inter ſe, cognitarum et incognitarum, conſtantium et variabilium, relatio, quomodo pendeant a ſe invicem, determinarique poſſint. Tantum autem abeſt, ut formulae per Analyſin quaeſitae ſemper ſint, pro rei quidem ipſius modo atque natura, faciles ſatis et ad uſum aptae, ut potius, praeſertim, ubi tractandae proponantur Series, quae Analyſeos conſtituunt partem longe maximam et fructuoſiſſimam, difficillimae haud raro et fere intractabiles reperiantur, modus etiam interdum, quo ſolutiones, quae expetuntur, abſolvi et ad finem perduci poſſint, non appareat. Cuius quidem difficultatis videmur nobis duplicem potiſſimum cauſam obſervaſſe. Prima eſt, quod ea, quae natura ſua atque ipſa re deprehenduntur ſeiuncta, tractantur plerumque coniuncta; dum coefficientium v. c. et exponentium, quantitatum determinatarum et indeterminatarum viciſſitudines in ſingulis terminis et varietates, una quaerunt eademque opera *omnes ſimul* indagare, una formula, quae Legem progreſſus contineat, omnes complecti (a); quo fieri non poteſt, quin formula quaeſita evadat ſaepe-

(a) *Poſſunt* quidem, atque adeo etiam *debent*, ſingulorum terminorum varietates et viciſſitudines una eademque formula generali exhiberi, ſed *expedit, eas non omnes quaerere ſimul, ſed per partes*, quae deinde ſingulae, ſuo quaeque loco ac ordine, uniantur. Verbo, imitari debemus Theorematis Binomialis Analyſin, uti vulgo inſtituitur, qua Exponentium et Coefficientium, ubi prius ſeparatim quaeſita forma eſt, termini deinde, qui ſibi reſpondent, coniunctim ponuntur in formula; quae cautio eo magis adhibenda eſt, quo plures ſunt partes.

Sic

ſaepenumero complicatiſſima, quae, ubi res ad uſum converti debet, ad ſubſtitutiones, propter terminos implicitos, cogat nos multiplices et reductiones plane taedioſas recurrere, praeſertim, ubi formula Methodo Serierum Leibnitiana, per ſuppoſitionem Seriei quaeſitae tanquam inventae, efficta, terminos producit conſtanter ſequentes per antecedentes. Altera cauſa eſt, quod ratio in his formulis habetur nulla *Combinationum* et *Permutationum*, quae in quantitatum ſingularum, veluti partium, ex quibus termini conſtant, Coniunctione apta, earumque inter ſe ordinis ac ſitus Variatione legitima poſitae ſunt; cum tamen formulae, quae hic quaeruntur, generales, ut alibi iamiam monui et res ipſa loquitur, nihil aliud doceant, quam quo ordine quantitates datae ſint colligendae, h. e. *combinandae* vel *permutandae*; quae ſitu etiam ſuo harmoniam quandam ſervant ſemper, per ſolutionis operationes explicabilem, et propter facilitatem, qua terminorum inde poſſunt ac debent coniungi ac variari partes ſingulae, ſummopere amplectendam. Quae cum animadvertiſſem, operam dedi, ut, quae ab *Arte Combinatoria* in

Sic v. c. pro Serie $ay + by^2 + cy^3 + dy^4 + ey^5 \ldots$ in Seriem transformanda hujusmodi

$$\frac{\mathfrak{A}y}{(\alpha+\beta y)} + \frac{\mathfrak{B}y^2}{(\alpha+\beta y)^2} + \frac{\mathfrak{C}y^3}{(\alpha+\beta y)^3} + \&c.$$

(ubi $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$.... coefficientes fictos, pro ſingulis terminis determinandos ſignificant) habito reſpectu poteſtatum literae y, Binomiique $\alpha + \beta y$, et Coefficientium, a, b, c, d....et α, β, itemque Coefficientium binomialium 1, $\frac{n-1}{1}$, $\frac{n-1.\,n-2}{1.\,2}$, $\frac{n-1.\,n-2.\,n-3}{1.\,2.\,3}$, reperi, atque nuper oſtendi, termini n^{ti} indefiniti $\frac{\mathfrak{N}y^n}{(\alpha+\beta y)^n}$ Coefficientem $\mathfrak{N}$ conſtare ex

Dignitatibus	β^{n-1}	β^{n-2}	β^{n-3}	......	1
" " "	α^1	α^2	α^3	......	α^n
					$n-1$
et Coefficientibus	a	b	c	......	a
" " "	1	$\frac{n-1}{1}$	$\frac{n-1.\,n-2}{1.\,2}$	......	1

Serierum verticalium ſingulis terminis in ſingulos ductis, unde producitur

$$\mathfrak{N} = a\alpha^1\beta^{n-1} + \frac{n-1}{1}b\alpha^2\beta^{n-2} + \frac{n-1.\,n-2}{1.\,2}c\alpha^3\beta^{n-3}.\,2.\,+ \overset{n-1}{a}\,\alpha^n$$

ubi n^{tus} Coefficiens ex his a, b, c, d, e...... ſignificatur hic per $\overset{n-1}{a}$, qui ſcilicet $n-1$ locis diſtat a primo a. Aliam problematis hujus ſolutionem, per terminos recurrentes, attulit *Eulerus Inſtitt. Calc. Diff. P. II. Cap. VIII.*

in Analyſi poſſint, non expectari modo, ſed percipi etiam fructus, liquido appareret, edito nuper libello (b), in quo Methodum tradidi univerſalem, propoſitis etiam, ut applicatio oſtenderetur, pluribus exemplis, quae Methodi bonitatem et opportunitatem, formularumque, inde provenientium, analytico-combinatoriarum concinnitatem atque uſum ſatis, opinor, probavere. Supereſt, ut, quae in eo libro breviter tantum dicta ſunt, et tumultuarie quodammodo propoſita de Methodo (namque Infinitinomii Dignitatibus, quibus eſt fere propria, primo loco applicabilem tantum eſſe exiſtimaveram, poſt tamen, et currente iamiam prelo, univerſalem reddi et quibuslibet adeo Seriebus adhiberi eam poſſe cognoveram) eiusque in Analyſi uſu, fuſius atque ordine explicentur. Abſolvi autem, quod quaeritur, commode poterit duplici Opere, ſive eiusdem operis duplici parte: Altera, *Variationum*, h. e. *Permutationum* et *Combinationum novo Syſtemate* propoſito, ſic, ut *Variationum* non ſolum *Numerus* et multitudo formulis definiatur (in quo multi iamiam fuere, LEIBNITIUS (c) et JACOBUS BERNOULLIUS (d) Viri Summi) ſed, quod praecipuum eſt in hac re, quodque Analyſin Artesque reliquas et Scientias plurimum iuvat, *ipſae* etiam *Formae* (e), ſive *Species* exhibeantur, quae fere hactenus neglectae iacuere, numerorum potiſſimum ſubſidio facillime eruendae: Altera *Logiſtica Serierum*, in qua problematum ac theorematum prioris, veluti *Analyſeos novi Organi*, ad quasvis Series, et per has ad omnem Analyſin applicatio inſtituetur.

Utriusque partis exhibebimus hic, ſpeciminis loco, particulam: illius quidem, *Verborum Definitiones*, veluti principia, additis Permutationum, Combinationum, Variationum, Propoſitionibus, quae ad *formas* ſpectant, paucis illis quidem, ſed maxime univerſalibus, una cum formulis, quibus *Numerus formarum* ſive *ſpecierum* exponitur; huius autem *Conſpectum Tabularum*,

(b) *Infinitinomii Dignitatum — Hiſtoria, Leges ac Formulae — Methodus Potentiarum — et Serierum ab Evolutione Factorum quotcunque oriundarum Geneſes; cum Appendice X Tabularum Combinationum, Permutationum ac Variationum. Gottingae apud I. C. Dieterich 1779.*

(c) *Diſſertatio de Arte Combinatoria*, in qua ex Arithmeticae fundamentis Complicationum ac Transpoſitionum Doctrina novis praeceptis exſtruitur, et uſus ambarum per univerſum ſcientiarum orbem oſtenditur, nova etiam Artis meditandi, ſeu Logicae Inventionis ſemina ſparguntur — *Lipſiae 1666.* repetita Francofurti, et *Leibnitii Opp. T. II. P. I. p. 339-399. inſerta a Lud. Dutens.*

(d) *Ars conjectandi; Opus poſthumum. Baſileae 1713.*

(e) Tentarunt quidem ſubinde etiam *formas Leibnitius* et *Bernoullius*, ſed breviſſime et eo tantum conſilio, ut, quos fere unice quaerebant, *numeros formarum* inde deducerent.

bularum, unde res ipſa, qualis futura ſit perſpici poſſit. Verborum in hac materia, vel iam antiquitus frequentatorum, vel a LEIBNITII demum inde BERNOULLIIQUE temporibus uſu receptorum ſignificationes, non temere mutavimus, ſed potius, tollendo etiam quorundam ambiguitatem, confirmavimus, additis novorum verborum, ſi quidem res ipſa poſtulare videretur, novis definitionibus.

§. II.

Verborum Definitiones.

1. ARS COMBINATORIA docet exhibere atque enumerare modos poſſibiles omnes, ſecundum quos res plures propoſitae coniungi, transponi et permiſceri poſſunt.

Res propoſitae ſpectantur hic ut partes, vel minimae, vel pro minimis ſuppoſitae, unde Tota plura (partibus vel omnibus, vel quibusdam) diverſa, tanquam ex Elementis, naſcuntur. *Fundamentum* Artis combinatoriae ſunt *Variationes*, ſive relationis mutationes, quarum duo ſunt genera: *Complexionis* alterum, quo partes datae varie componuntur; alterum *Permutationis*, quo compoſitarum partes varie diſponuntur. Hinc rerum, quae ſpecie differunt, diverſitas ſecundum partes vel alias, vel aliter diſpoſitas, h. e. ſecundum vel *materiam*, vel *formam*. Complexiones igitur et Permutationes ſumma duo Artis Combinatoriae Capita conſtituunt; quibus iungitur tertium, in quo Complexiones ſimul cum Permutationibus, h. e. Variationes pertractantur.

2. NUMERUS RERUM coniungendarum transponendarum definitur rerum, ſive partium datarum multitudine.

Hunc dixit *Numerum* ſimpliciter LEIBNITIUS. Idem Combinatoriam Artem maluit *Complicatoriam* (quia non omnis Complexus proprie Combinatio, ſive rerum duarum coniunctio eſt) et Combinationes *Complicationes* appellare.

a) Permutationes ſive Transpoſitiones.

3. PERMUTATIONES RERUM quocunque numero vocantur, cum datae res, ſervata earum multitudine, ſed non ordine, omnibus, quibus poſſunt, modis coordinantur et transponuntur.

Sic, poſitis tribus v. c. rebus ſive literis *a*, *b*, *c*, variando ipſarum ordinem, Permutationes prodeunt, ſive Transpoſitiones:

abc; acb; bac; bca; cab; cba;

Literas

Literas nempe hic aſſumimus pro rebus datis quibusque; quia optime quid ſit, quod quaeritur, explicant, et omnibus rebus pari analogia faciliime applicari poſſunt. Expedit tamen haud raro pro literis ſcribere numeros; ut ſuo loco oſtendemus.

4. RES REPETITA dicitur, quae in eadem Variatione ſaepius ponitur.

Nempe res permutandae vel *omnes* ſunt *diverſae*, *a*, *b*, *c*, *d* . . . vel *aliquot* earum *eaedem*, veluti *a*, *b*, *b*, ubi res repetita *b*, unde Permutationes:

abb; bab; bba;

5. FORMA PERMUTATIONIS eſt rerum omnium transponendarum poſitio.

6. NUMERUS PERMUTATIONUM dicitur, qui formas numerat ſive ſpecies Transpoſitionum diverſas.

Sic Permutationum a tribus rebus diverſis, *a*, *b*, *c*, numerus eſt 6. [3]; a rebus autem *a*, *b*, *b*, quarum una [*b*] ponitur bis, eſt 3, [4]. LEIBNITIO *Numerus* hic dicitur *Variabilitas Ordinis*.

7. SITUS refertur ad loca, quae rerum datarum (*Totius*) ſingulae (*Partes*) occupant.

Significanter itaque LEIBNITIO *Situs* dicitur *localitas partium*.

8. SITUS ABSOLUTUS, ſive *Partium* cum *Toto*, reſpicit numerum locorum et diſtantiam, qua removentur ab extremis.

Refert itaque hic, quo quaeque res *loco* ponatur; *primo*, *ſecundo*, *tertio* etc.; quae ſtet *priori*, quae *poſteriori*. Mutatis rerum ſedibus, mutatur etiam Situs abſolutus.

9. SITUS RELATUS, ſive *Partium* ad *Partes*, ſolam reſpicit Partium inter ſe antrorſum retrorſum diſtantiam.

In *ſitu relato* termini extremi *contigui* intelliguntur. Quapropter appoſite ſolent res, quarum proponitur ſitus relatus *figurate* diſponi literis, circuli in modum (vel aliarum linearum figuram claudentium, polygonorum etiam regularium) in orbem redeuntibus. Similiter poſſunt res in *ſitu abſoluto figurate* poni, vel ſecundum curvarum hiantium (quae ſpatium non concludunt) vel etiam, quod ſimplicius eſt, ſecundum lineae rectae tractum.

10. ORDO RERUM cenſetur Situ abſoluto, VICINITAS relato; ille DISPOSITIO, hic COMPOSITIO interdum dicitur.

Termini ſunt *Leibnitiani*, quibus ſubinde ſynonymice utitur. Sic partium *a*, *b*, *c*, *d*, *e*, ſitus ſequentes:

 abcde;

abcde; bcdea; cdeab; deabc; eabcd;

Ordine differunt vel *Diſpoſitione*, non autem *Vicinitate* vel *Compoſitione*.

11. CAPUT PERMUTATIONIS eſt partium, ex pluribus quae permutandae ſunt, certarum, etiam unius, poſitio.

Quaelibet res vel litera, ſive per ſe, ſive cum alia aut aliis, contiguis vel non contiguis, h. e. rerum quaelibet coniunctio, continua ſeu discontinua, quae tota continetur in alia plurium literarum coniunctione, huius dici poteſt *Caput*. Sic formae huiusmodi: *acfhkpq Capita* conſtitui poſſunt res quaelibet, vel *ſingulae*, *a*, *f*, ... vel *binae; ac*, *hq*.... vel *ternae*, *fhk*, *akq* .... et ſic porro.

12. PERMUTATIONES COMMUNES ſunt, in quibus plura Capita concurrunt.

13. CAPUT MULTIPLICABILE dicitur, cuius poſſunt partes variari.

14. RES HOMOGENEA eſt, quae dato loco aeque poni poteſt, ſalvo Capite. MONADICA, vel ſolitaria, quae homogeneam non habet.

Verba, quorum poſuimus in 11-14 Definitiones, primus adhibuit in Arte Combinatoria LEIBNITIUS, praetermiſit BERNOULLIUS, ut alia multa.

β) *Combinationes ſive Complicationes.*

15. COMBINATIONES RERUM vocantur, ubi, poſitis rebus pluribus, ſive diverſis omnibus, ſive aliquot earum iisdem, iunguntur quotlibet binae, ternae, quaternae, et ſic porro, quoad eius fieri poteſt, nullo tamen Ordinis ſingularum Viciſſitudinisve habito reſpectu.

Cavetur hic, ne poſſint, ut fit in Permutationibus [3], coniunctiones, ex iisdem nomine rebus et ſolo Ordine diverſis, conflatae, ſaepius quam ſemel in Combinationibus occurrere. Ceterum Combinationes hic dicuntur ſenſu latiori, pro *Complicationibus* ſive Coniunctionibus rerum quocunque numero, non tantum duarum, de quibus Combinationes dicuntur proprie et ſecundum *ἔτυμον*, ſenſu ſtrictiori.

τὸ ἔτυμον origo vocabuli.

16. EXPONENS ſive INDEX COMBINATIONIS dicitur Numerus, quo definitur, quot res ex datis coniungi ſimul debent.

Itaque poſitis rebus pluribus, *a*, *b*, *c*, *d* ..... *binarum* quotlibet: *ab*, *ac*, *ad*, *bc*, *bd*, *cd* .... *Exponens* eſt 2; *ternarum: abc*, *abd*, *acd*, *bcd* .... *Exponens* eſt 3; *quaternarum: abcd* .... *Exponens* eſt 4; ſimiliterque pro reliquis Exponentibus, ſecundum rei ſive quaeſtionis cuiusque modum atque naturam, adhibendis vel ſingulis, vel omnibus ſimul. Hinc rerum datarum Combinationes ſecundum *Exponentes* vel *ſingulos ſeorſim*, vel *omnes coniunctim*.

17. BINIONES, TERNIONES, QUATERNIONES, QUINIONES, SENIONES, et reliquae COMPLEXIONES, interdum etiam BINARII, TERNARII, QUATERNARII, QUINARII, SENARII etc., dicuntur ipſae res, ſecundum Exponentes 2, 3, 4, 5, 6 etc. iunctae, quae ſimul ut unum conſiderantur. Hinc UNIONES, ubi ſingulae, NULLIONES, ubi nullae ponuntur res, h. e. quarum ſunt Exponentes 1, vel 0.

Scilicet *Uniones* et *Nulliones* (pro quibus LEIBNITIUS uſurpavit *Olliones*) ad reliquorum ſunt Analogiam efficti. *Uniones* ſunt a, b, c etc. *Biniones:* aa, ab, bb, bc etc. *Terniones:* aaa, aab, abc; etc.

18. COMBINATIONES, CONTERNATIONES, CONQUATERNATIONES, CONQUINATIONES etc. verbo: COMPLICATIONES rerum datarum, nominantur binarum, ternarum, quaternarum, quinarum etc. in Biniones, Terniones, Quaterniones, Quiniones et in reliquas Complexiones, Coniunctiones.

LEIBNITIO *Complexio* dicitur *Unio minoris Totius in maiori;* BERNOULLIUS contra abſtinere ſe ab huius verbi uſu, et Complexiones appellare maluit *Electiones*, exiſtimans, ſic melius ſubintelligi poſſe, eas etiam rerum acceptiones, ubi vel *ſingulae* ſeorſim, vel *nullae* plane ſumantur. Sed tamen recte habet verbum Complexionis, hoc etiam ſenſu, *καταχρηστικῶς* quidem, ſed tamen ſecundum *ἀναλογίαν* acceptum, ubi praeter Exponentes 2, 3, 4, 5 .... admittuntur etiam 0 et 1 [17]. Solent autem verborum iſtorum *quantitates* ſignificanter exprimi per ipſos numeros Exponentes, ſcribendo: *2niones, 3niones* etc. *2narii, 3narii* etc. *Com2nationes*, *Con3nationes*, etc.

abuſive

19. CLASSES COMPLEXIONUM colliguntur ex rerum datarum Coniunctionibus, et a Complexionum Exponentibus denominantur: *prima*, ſi Exponens eſt 1; *ſecunda*, ſi eſt 2; *tertia*, ſi 3; et ſic porro; Exponentibus, et quae hinc pendent, Claſſibus, in Infinitum haud raro pergentibus.

Hoc fit, ubi rerum ponitur combinandarum multitudo, adeoque etiam Complexionum in ſingulis Claſſibus, numero quolibet finito maior; vel, ſi etiam finita ſit rerum multitudo, res tamen repetitae [4] admittuntur, quocunque numero, in Complexionibus. Sic poſitis *tribus* rebus *a, b, c*, veluti *1nionibus*, in *prima* Claſſe; *ſecunda* Claſſis conſtat *Com2nationibus*, a 2nionibus *aa, ab, ac, bb, bc, cc; tertia* Claſſis *Con3nationibus*, a 3nionibus: *aaa, aab, aac, abb, abc, acc, bbb, bbc, bcc, ccc;* ſimiliterque *quarta, quinta, ſexta* .... *n*ta Claſſis, *Con4nationibus*, *Con5nationibus*, *Con6nationibus* ..... *Connnationibus*, a 4nionibus, 5nionibus, 6nionibus, ... *n*nionibus oriundis, reſpective. Hinc *Complexiones*, *admiſſis* vel *neglectis repetitionibus*, dicuntur.

20. ORDINES

27. COMPLEXIO UTILIS diſtinguitur ab INUTILI; illa pro ſubiecta materia, ſive problematis conditionibus aut determinationibus, locum habet, haec non habet. Idem intellige de PERMUTATIONIBUS UTILIBUS et INUTILIBUS.

Sic literarum *a*, *b*, *c* .... *Terniones* ſunt, admiſſis repetitionibus: *aaa*, *aab*, *aac*, *abb*, *abc*, *acc*, *bbb*, *bbc*, *bcc*, *ccc* .... [19]. Quod ſi igitur *a unius*, *b duarum*, *c trium*, *Dimenſionum* Factores deſignent, et aliunde conſtet, Formulae, quae quaeritur, terminos ſingulos *ſeptimam* v. c. ſuperare *Dimenſionem* non poſſe: intercidunt hic Claſſis *tertiae* Complexiones *bcc*, *ccc* .... cum reliquis, reliquarum etiam Claſſium, quae altius aſcendunt; *ſeptimae* autem Claſſis *prima* Complexio *aaaaaaa* terminum ultimum conſtituit.

Sic vocis *Roma* ἀναγραμματισμοὶ ſignificantes Permutationibus continentur *utilibus* his: *amor*, *armo*, *Maro*, *mora*, *oram*, *ramo*; reliquae, ſeptendecim numero, *inutiles* ſive nudi ſunt, latinis quidem auribus, et ſine mente ſoni HAX. LAX. MAX. DEAX. ADIMAX, et ABRACADABRA παλινδρομον.

recurrere.

28. COMPLEXIONES DATI CAPITIS dicuntur, quas ingreditur Caput.

29. COMPLEXIONES COMMUNES SUNT, in quibus plura Capita concurrunt.

De *Capite* vid. ad Defin. 11. Capitis ſcilicet potiſſimum rationem habemus cum reſpectu ad reliquarum partium vel *transpoſitionem* in Permutationibus, vel *coniunctionem* ex datis in Complexionibus, v. c. ubi poſitis *ſubiectis* quaeruntur praedicata, *praedicatis* ſubiecta; Subiecta aut Praedicata Capita conſtituunt.

30. COMPLEXIONES NUMERORUM ſive NUMERICAE ſunt, quae fiunt poſitis numeris pro rebus ipſis. Idem intellige de PERMUTATIONIBUS NUMERORUM.

A *Complexionibus* itaque et *Permutationibus literarum* re ipſa non differunt. Nam ſive pro rebus datis

ponantur literae *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, ... ex ordine
ſive numeri 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ... reſpective,

facile patet, numeros eodem modo, quam literas combinari poſſe atque permutari, poſitis v. c. 1, 2, 3 pro *abc*; 3, 3, 3, 6, 6, 10 pro *cccffk*; $2^m 7^n 10^r 11^s$ pro $b^m g^n k^r l^s$; numeris ſcilicet 2, 7, 10, 11 ſingulis tot vicibus, quot ſunt in *m*, *n*, *r*, *s* Unitates, reſpective.

Refert autem ſaepiſſime, *numeros* ponere pro rebus, non literas; quia numeri, ut iam LEIBNITIUS notavit, rerum, praeſertim ubi magna eſt varietas, optime et ſimpliciſſime indicant relationes, quas habent ex prima ſuppoſitione inter ſe, quarum literae, ob vagam ſuam generalitatem, mente-

tem non, uti par eſt, admoneat. Quapropter etiam problematum ſive theorematum *Canonibus*, formularum analytico-combinatoriarum ope per Tabulas exhibendis, *Indiculum* addemus conſtanter, ſimilem ei, quem hic poſuimus, unde conſtet, quae literae (ſive res) pro numeris ſingulis Complexionum aut Permutationum ſint ſubſtituendae.

31. SECTIONES NUMERI DATI fiunt, colligendo ex numeris minoribus, tanquam partibus dati maioris, tot, quot, ſumti ſive additi ſimul, numero dato ſunt aequales.

LEIBNITIO nominantur *Diſcerptiones (Zerfaellungen)* quales v. c. in *praxi* ſic dicta *Italica* ſubinde adhibentur. Hinc

32. COMPLEXIONES NUMERI PROPOSITI *n* dicuntur ſectiones poſſibiles omnes, unde numerus *n* producitur.

Hoc fit *addendo* quoslibet *binos*, *ternos*, *quaternos*, *quinos*, *ſenos* etc. numeros (ſive ſint *inaequales* omnes, ſive etiam *aliquot* eorum *aequales*) ex ſerie naturalium 1, 2, 3, 4, ... *n*-1, quorum ſimul *ſumma* aequatur numero propoſito *n*. Sic v. c. Numeri 5 *Complexiones* conſtant *rite ordinatas* [25] in *Claſſe*

prima ex numero propoſito ipſo 5
ſecunda ex numerorum *Com2nationibus* 1, 4 et 2, 3
tertia - - - - - *Con3nationibus* 1, 1, 3 et 1, 2, 2
quarta - - - - *Con4natione* 1, 1, 1, 2
quinta - - - - *Con5natione* 1, 1, 1, 1, 1

unde patet ſimul, tot ſemper Claſſes eſſe huiusmodi Complexionum, quot ſunt Numeri *n* Unitates.

Terminorum ex partibus, quae *ſummam* efficiant *datam*, uſum oſtendimus (*h*) in Analyſi, Serierum potiſſimum tractatione, longe frequentiſſimum

(*h*) *Infinit. Dignit.* §. XXIII. Hinc *Methodus* noſtra *Potentiarum* §§. XXIV, XXV, XXVI, et *Serierum ex plurium Complicatione Geneſis* §. XXVII. Fundamentum ſunt *Sectiones numeri dati* [31] ſive *Diſcerptiones*, quas quidem tentavit, ſed non abſolvit, *Leibnitius*; namque, *ubi plures partes admittuntur, ingens panditur abyſſus diſcerptionum.*" *Diſſert. de Art. Combin. Probl. III. Sect.* 25. Huc pertinent duo problemata Combinatoria generaliſſima, quarum dedi ſolutionem §§. XXII et XXVII. l. c. Idem *Leibnitius* triginta amplius annis poſt editam Diſſertationem ſuam de Arte Combinatoria quaerit ex *Ioh. Bernoullio*: *An unquam conſideraſti numerum diſcerptionum, vel divulſionum numeri dati, quot ſcilicet modis divelli poſſit in partes duas, tres etc. Videtur mihi ejus determinatio non facilis et tamen digna, quae habeatur. Commerc. Epiſtol. Leibn. et Bern.* T. I. p. 462. *Numerum* ſpecierum omnium, numero propoſito aequalium, ſed non *ſpecies* ipſas, ſive *formas*, primus exhibuit *Eulerus*, tabulae ampliſſimae ſubſidio: *Nov. Comment. Petrop. T. III. p. 125. ſſ.*

tiſſimum eſſe. Summa autem huiusmodi, per literas vix ac ne vix quidem, *numerorum* autem ope *facillime*, obtineri poteſt; quo fit, ut *Numeri definiti Complexiones*, quae partem nobis Elementorum combinatoriorum conſtituunt, maximi ſint momenti.

33. COMPLEXIONES DUCUNTUR IN COMPLEXIONES, ubi, propoſitis Complexionibus pluribus, harum ſumuntur ex ordine, eodem modo quo rerum ſingularum, Complexiones. Idem intellige de PERMUTATIONIBUS COMPLEXIONUM, quarum tantum ordo ſitusve variari debet.

Sic poſitis *Complexionibus* $\overset{a}{\alpha};\ \overset{b}{\beta\gamma};\ \overset{c}{\gamma^2\delta};\ \overset{d}{\delta^2\varepsilon^2};$ ..... prodeunt inde *Com2nationes*: $\alpha.\beta\gamma$; $\alpha.\gamma^2\delta$; $\alpha.\delta^2\varepsilon^2$; $\beta\gamma.\gamma^2\delta$; $\beta\gamma.\delta^2\varepsilon^2$; $\gamma^2\delta.\delta^2\varepsilon^2$; ..... *et Con3nationes*: $\alpha.\beta\gamma.\gamma^2\delta$; $\alpha.\beta\gamma.\delta^2\varepsilon^2$; $\alpha.\gamma^2\delta.\delta^2\varepsilon^2$; $\beta\gamma.\gamma^2\delta.\delta^2\varepsilon^2$; ..... cum reliquis *Complexionum Complexionibus*, eo modo, quo rerum ſingularum *a*, *b*, *c*, *d* .... Complexiones [in 15].

Propoſitae Complexiones plerumque complectuntur res ſingulas unius eiusdemque Seriei, interdum tamen res ſive literae, in Complexionibus poſitae, ad plures Series pertinent.

34. CLASSES DUCUNTUR IN CLASSES, ubi, propoſitis plurium Serierum, ſive etiam unius, Claſſibus Complexionum, iunguntur quaelibet Claſſes (vel Complexiones ex Claſſibus) vel *binae* ex *duabus*, vel *ternae* ex *tribus*, vel *quaternae* ex *quatuor*, et ſic porro, Claſſibus (ſive Claſſium Complexionibus) ſic, ut in qualibet Complexione ex qualibet Claſſe (vel Claſſis Complexione) Complexio una appareat.

Hinc *Claſſium* (ab una vel pluribus Seriebus) *Complexiones*. Sic poſitis Claſſibus ′A, ′B, ′C, etc. ſive ad *unam* eandemque Seriem ſingulae ſive ad *diverſas* pertineant

(′A) *a*, *b*, *c*, *d*, *e* .... (′B) 𝔞, 𝔟, 𝔠, 𝔡, 𝔢 .... (′C) $\alpha, \beta, \gamma, \delta, \varepsilon$ .... (etc.), (ubi literae ſingulae quaslibet poſſunt deſignare *Uniones*, *Biniones*, *Terniones* .... verbo, *Complexiones*) *Complexiones Claſſium* inde fiunt, ſive *Claſſes ducuntur in Claſſes*, ductis ſingulis Complexionibus *unius* Claſſis in ſingulas *alterius*, et quod inde producitur in ſingulas *tertiae*, et factum ex his in ſingulas *quartae* Claſſis Complexiones; et ſic porro, quam diu Claſſes extiterint.

Incidit LEIBNITIUS etiam in *Complexiones Claſſium* a pluribus Seriebus (i), ſed *Claſſium* tantummodo *primarum*. *Superiorum* Claſſium ex pluribus Seriebus *Complexiones* non attigit, neque etiam *Complexiones ſuperiorum Ordinum*, de quibus mox videbimus.

35. COM-

(i) Diſſert. de Arte Comb. Probl. III, ſ. 1. Opp. T. II. p. 376.

35. COMPLEXIONES PRIMI ORDINIS dicuntur, quae fiunt a rebus ſingulis *a*, *b*, *c* etc.; COMPLEXIONES SECUNDI ORDINIS, quae fiunt a Complexionibus primi Ordinis. Hinc TERTII, QUARTI, QUINTI ... *n*TI ORDINIS COMPLEXIONES, quae fiunt a ſecundi, tertii, quarti .... (n-1) ti Ordinis Complexionibus, reſpective; pergentibus nimirum Complexionum hoc modo Ordinibus in infinitum.

Hinc CLASSES *Complexionum* primi, ſecundi, tertii, quarti etc. Ordinis, PRIMA, SECUNDA, TERTIA, QUARTA etc.

Complexiones propoſitae ſingulae, vel integrae *Claſſes*, quarum poſtulantur Complicationes aut Permutationes, vel ad *unam* eandemque Seriem primitivam *omnes*, vel ad Series *diverſas* pertinent, adeoque varietati, verbis nullo modo aſſequendae, porta patet hic ampliſſima. Et tamen haec tanta varietas numeris coerceri poteſt, ſine quibus exitus daretur nullus; praeſertim, cum etiam hic *Complexionum numeri propoſiti n* ampliſſimus uſus ſit in Problematis, propter terminorum et multitudinem et ποικιλιαν, maxime impeditis.

γ) *Variationes ſive Complicationes cum Permutationibus.*

36. VARIATIONES RERUM datarum dicuntur Complexiones earum ſimul omnes cum Permutationibus.

Nam praeter *Complexionum* varietatem, *ordinis* etiam interdum, *ſitusve* partium ſimul habenda ratio eſt, ut fieri v. c. ſolet in *numeris* et *ſyllabis* verborum, ſyllabarum literis cum ipſis ſyllabis tranſpoſitis.

Differunt autem hic etiam *Variationes rerum datarum*, h. e. rerum Complexiones cum Permutationibus, a *Variationibus Claſſium*, ſive Claſſium Complexionibus cum Permutationibus; itemque *Variationes Numerorum datorum*, h. e. Numerorum Complexiones cum Permutationibus, a *Variationibus numeri propoſiti n*, quibus numeri *n* exhibentur ſimul Complexiones cum Permutationibus. Neque etiam *Variationes*, ſive *Complexionum ſingularum*, ſive *Claſſium integrarum*, ſed *Ordinum ſuperiorum*, quid ſint, poterit obſcurum eſſe. Namque licet *hic* brevioribus eſſe nobis propter ea, quae de Permutationibus et Complexionibus ſeorſim diximus fuſius in ſectionibus antecedentibus duabus α et β.

Pro eo, quod nos dicimus: *Variationes quaerere*, LEIBNITIUS dixit: *Variationes ſitus*, *ſeu diſpoſitiones*, *ducere in Complexiones.*

δ) *Operationes combinatoriae. Elementa combinatoria. Methodus. Logiſtica Serierum.*

37. OPERATIONES COMBINATORIAE ſunt ipſae rerum Permutationes, Complicationes, Variationes, et quae ex hi. utcunque componuntur.

38. ELEMENTA COMBINATORIA dicuntur datae ipſae res, et quae fiunt ab his Permutationes et Complexiones, vel ſingulae, vel collectae in Claſſes; quatenus res alias, quae ab iis pendent, ingrediuntur, veluti partes conſtitutivae.

Elementa vere diximus, quorum ſcilicet, utpote *minimorum*, vel pro minimis *ſuppoſitorum*, coniunctione varia vel diſpoſitione, res compoſitae oriuntur, ſive *materia*, ſive *forma* diverſae a ſe inuicem. Uſus horum Elementorum praecipuus poſitus in eo eſt, quod rei quaeſitae *partes conſtitutiuas* exhibeant facillime, ſive *ex ordine* quaerantur, ſive *extra ordinem;* quod *Legem progreſſus* in his partibus exponant ſumma cum perſpicuitate; quod partes ponant ſtatim eo *loco*, quo manere, eaque *forma*, quam ſeruare debent, declinatis *ſubſtitutionum* et *reductionum*, haud raro taedii pleniſſimarum, *diverticulis*.

39. METHODUS COMBINATORIA docet, Elementa quomodo coniungi, transponi ac permiſceri debent, ut Compoſitum inde, quod quaerimus, reſultet.

Adhibenda igitur ſimul *diſciplina* eſt, ad quam partes variandae, ſive Totum, quod inde volumus componere, pertinent. Complicationum ac Permutationum uſum editis ſpeciminibus ex univerſo ſcientiarum orbe, oſtendit LEIBNITIUS (k), et poſt eum IACOBUS BERNOULLIUS (l) et NICOLAUS BERNOULLIUS (m). Sed *Methodus* ipſa tamen adhuc deſideratur *perfecta* et *univerſalis*, quae omnibus aeque rebus applicari poſſit; quae doceat vel per *praecepta*, vel *formularum* ope *quomodo* agenda ſunt, ſimul ac ex diſciplina, ad quam pertinent res, de quibus quaeritur, conſtiterit, *quae* ſint agenda. Scilicet LEIBNITIUS, rem aggreſſus admodum iuvenis, ſibi ipſe non ſatisfecit (n). Explicare rem ulterius praeceptis, promove-
re

(k) *In Diſſert. de Arte Combin.* propoſitis XII. Problematis, quibus ſingulis varia theoremata adiunxit, varios uſus.

(l) *In Arte conjectandi*, *ſive Stochaſtice. Baſileae* 1713. P. III. explicat uſum Artis conjectandi in variis *Sortitionibus* et *ludis;* P. IV. Uſum explicat et applicationem in *Civilibus*, *Moralibus*, *Oeconomicis;* ſed hanc partem reliquit imperfectam. Huc pertinet etiam, *The doctrine of Chances; by A. de Moivre, Edit. ſec. Lond.* 1738.

(m) Editor Artis conjectandi, poſt mortem Auctoris, patrui ſui. Specimina Artis combinatoriae ad *Ius applicatae* propoſuit in *Diſſertatione inaugurali. Baſileae* 1709. Diſſertationem hanç, cujus meminit Auctor ipſe in praefat. ad *Iac. Bern.* Artem conject. et *Ioh. Bern.* in Epiſtola ad *Leibnitium. Comm. ep. T. II. p. 217.* videre mihi nondum contigit.

(n) Huc pertinent verba *Leibnitiana: Combinatoriam, quam animo complexus ſum, ex ea, quam pene puer conſcripſi, et anno 1666. edidi (me inconſulto ante annos aliquot recuſam) nolim aeſtimari; tametſi iam tum quaſdam meditationes*

re ſpeciminibus, ad finem perducere, ſublatis, quae moram iniicere poſſint, difficultatibus, promiſit quidem ſaepius (o), ſed multis graviſſimisque negotiis aliis praepeditus, manum IPSE admovere Operi non potuit; IAC. BERNOULLIUS, morte praeventus, Opus reliquit nondum abſolutum; NICOLAUS denique BERNOULLIUS applicationem tantum ad Iuris quaedam Capita inſtituit. Omnes autem id potiſſimum egerunt, ut *ſpecierum* in Permutationibus et Combinationibus *Numeros* inveſtigarent; cum Methodus ipſas potius *Species*, ſive *Formas* deſideret. Sed *Methodus* videtur *univerſalis* contineri, in ea, quae mihi

40. LOGISTICA SERIERUM dicitur, quae rerum ex datis combinandarum permutandarum tradit modum ac rationem per formulas analytico-combinatorias.

Nempe *rerum datarum ſingularum complexus Series hic dicitur; formulae analytico-combinatoriae* praecepta exhibent, per ipſa Elementa combinatoria, ſed *analytice* expoſita: nam *ſedes Artis combinatoriae Arithmetica eſt.* Hinc *Serierum Logiſtica analytica,* quae aeque late patet in omni re combinatoria, quam ipſa Analyſis reſpectu quantitatis, diſcretae et continuae, cuius eſt Inſtrumentum maxime univerſale. Applicatio igitur, ſive etiam coniunctio Artis combinatoriae cum Analyſi tam prona, tamque arcta eſt, ut ne geodaeſia quidem aut trigonometria cum geometria arctiori vinculo coniuncta ſit. *Ars itaque Combinatoria*, *analytice* expoſita, Philoſophis aeque et Geometris utilis. Eius maximam partem iamiam abſolvimus.

§. III.

Operationes combinatoriae. Specierum ſingularum cum Numero Specierum brevis expoſitio et uſus.

1. Poſitis verborum Definitionibus, tanquam principiis, proximum eſt, ut, Operationes combinatoriae quomodo inſtitui et abſolvi poſſint, Elementa combinatoria quem habeant pro ſingulis formis, ſive etiam Claſſibus valorem, qui ſit Numerus ſpecierum, qui Elementorum uſus, nunc oſtendamus; breviter quidem, cum primas tantum lineas ducere hic, non tabulam ipſam exponere, animum induxerimus, ſed ita tamen, ut res ipſa, qualis ſit, liquido appareat.

a) For-

ditationes non poenitendas, nec momenti nullius adſperſerim. Opp. Leibn. *T. III. p. 367.* Haec ſcripſit Leibnitius triginta amplius annis poſt editam Art. Comb. anno 1700.

(o) Praeter loca ſuperius jam laudata, conf. quae attuli plura in praefatione libelli de Infin. Dignit.

α) *Formarum, ſive Specierum ſingularum inventio.*

2. Reduximus autem nunc totum negotium ad *ſimpliciſſimas numerandi Leges*, quibus utimur in *decadici, dyadici, tetradici, ſexageſimalis*, et aliorum quorumcunque *Syſtematum numeris ſcribendis;* quae ratio tam eſt expedita, tamque facilis (nam numeros ſcribere *ex ordine*, atque etiam, ubi opus eſt, *per ſaltus progrediendo*, quilibet poteſt) ut mirum videri poſſit, neque LEIBNITIUM (p), neque BERNOULLIUM (q), neque me ipſum (r), cum ſcriberem de Infinitinomii Dignitatibus et Methodo Potentiarum libellum, vidiſſe: *Operationum combinatoriarum fundamentum et veluti normam propoſitam eſſe in numeros ſcribendi modo ac ratione unam fere atque unicam*, h. e. *praeſtantiſſimam;* cum ipſi numeri, Permutationum, Combinationum et Variationum perfectiſſima et ſimpliciſſima omnis generis exempla (s) praebeant. Sed ita fit

(p) Quod ſi obſervaſſet *Leibnitius*, vidiſſet: *patere hic, non in abyſſum* [v. not. h.] *horribilem et pertimescendam, ſed in latam et jam ſtratam viam proſpectum.*

(q) *Iacobus Bernoullius*, quid ſit res, et quas habeat opportunitates, non videtur perſpexiſſe; namque nullum in ipſius Arte coniectandi exemplum numerorum pro literis poſitorum apparet. Neque etiam perſpexit *Ioh. Bernoullius*, quippe qui, cum ampliorem rei expoſitionem ipſi tranſmiſiſſet *Leibnitius*, reſcriberet: *Legi Tua de progreſſionibus: mihi nimis abſtracta et abſtruſa videntur. Habui etiam conſimiles ſpeculationes aliquando. Comm. Ep. T. I. p.* 470; ad quae *Leibnitius: Mea de Progreſſionibus, vel potius de Expreſſionibus obiter Tibi inſpecta puto; alioqui vidiſſes, nihil eſſe facilius, nihil naturae conſentaneum magis, patereque etiam Tironi jam ſum expertus. Ibid. p.* 473. *Cf. T. II. p.* 11, 13, 14. Sed *Leibnitius* Ipſe videtur hic in culpa eſſe. Nam Problema cum ſolutione, qua rem oſtendere volebat, eſt vere αἰνιγματῶδες, et propter abyſſum illam maxime, de qua modo diximus,

Horribile, infandum dictu, cui lumen ademtum.

Vid. quae pluribus de hac re diſputavimus in praefat. ad libellum de *Infin. Dignit.*

(r) Cum regulas traderem pro *Complexionibus numeri definiti* alteram, alteram pro *numeri definiti Variationibus. Infin. Dign.* §. XXII. *p.* 73. *ſeq. et* §. XXVII. p. 129 *ſſ.*

(s) Hoc monui iamiam p. 96. libelli, cui titulus: *Beſchreibung einer ganz neuen Art, nach einem bekannten Geſeze fortgehende Zahlen, durch Abzählen, oder Abmeſſen bequem und ſicher zu finden. Leipzig* 1776, *bey S. L. Cruſius.* Huius libri diſquiſitiones, quae inſtituuntur, numericae, *occaſionem praebuere et quodammodo materiem* alterius illius, de Infinitinomii Dignitatibus et Methodo Potentiarum. Namque numerorum *compoſitorum* factores, a *primorum* pendent atque inde determinantur *Complexionibus* rite ordinatis, Exponentium ex ordine 2, 3, 4, 5 .... Similis eſt ratio, reliquorum, qui ibi conſiderantur, numerorum. Permutationum ſcilicet et Complicationum campus tantum non eſt infinitus.

ſit plerumque; quae ſunt ante oculos non videmus, facilia negligimus, venamur difficilia.

Admiſſa autem hac ratione, praecepta, quae hinc deducuntur combinatoria, longe fiunt ſimpliciſſima et ad uſum maxime accommodata. Cui rei ut fidem faciamus ipſo facto, *tria* hic *Problemata* afferemus *generaliſſima*, in quibus totius rei cardo vertitur et continetur ſumma: pro *Permutationibus rerum datarum* unum; pro *Complexionibus* alterum; pro *Variationibus* tertium.

3. PROBLEMA I. *Rerum datarum a, b, c, d, e, f.... Permutationes exhibere poſſibiles omnes.*

SOLUTIO. Ponantur

pro *rebus* ipſis *a, b, c, d, e, f ... ex ordine,*
Elementa numerica 1, 2, 3, 4, 5, 6 ... *reſpective,*

Numerus, qui inde producitur *minimus* 123456 .... et *maximus* .... 654321, cum *omnibus intermediis*, qui *totidem* ſemper et *iisdem Elementis* conſtant, rerum datarum ſiſtent omnes Permutationes poſſibiles.

DEMONSTRATIO. Tranſpoſitionum ſpecies ne cogitari quidem, nedum exhiberi, ulla poteſt, cui non reſpondeat numerus ex Elementis pro rebus aſſumtis. Sit enim, ſi fieri poteſt, ... *fcadbe* eiusmodi ſpecies; ergo .... 631425 (Elementorum numericorum complexus) non eſt numerus. Q. E. A. Ergo omnes exhibentur Permutationes poſſibiles per hos numeros. Q. E. F.

4. EXEMPLUM. *Rerum quatuor abcd determinantur ex Elementis 1, 2, 3, 4, per numerum minimum 1234, et maximum 4321, cum intermediis, Permutationes ſequentes:*

abcd	*bacd*	*cabd*	*dabc*
1234	2134	3124	4123
..43	..43	..42	..32
.324	.314	.214	.213
..42	..41	..41	..31
.423	.413	.412	.312
..32	..31	..21	..21
adcb	*bdca*	*cdba*	*dcba*

Primo atque *ultimo* cuiusque Columnae numero adiunximus, exempli loco, ſpeciem tranſpoſitis literis reſpondentem. Puncta hic ſignificant figuras numericas, quibus reſpondent inferius poſita, et ſimul modum oſtendunt, quo poſſunt numeri, qui poſtulantur, 1234, 1243, 1324, 1342, etc. ſcribi

quam

quam celerrime. Nempe numeri *poſtremi duo* ponuntur conſtanter inverſo ordine (ex 34 fit 43, ex 24 fit 42, etc.) numeri autem a fine *tertii*, *quarti*, (eodemque modo, ubi plures ſunt numerorum ſeries verticales, etiam *quinti*, *ſexti*, etc) creſcunt ſucceſſive hac lege, ut figuram quamlibet 1, 2, 3, 4, quae poſita eſt in antecedentibus lineis ſuo loco, excipiat *proxime maior*, quam a dextra ſequantur in eadem linea, quae reliquae ſunt figurae, ſuo ordine ſingulae, incipiendo ſemper ab ea, quae minima eſt, progrediendo autem per reliquas proxime maiores; eodem fere modo, quo Claſſis definitae Variationes definitas deduximus (t).

5. Aliam Problematis Solutionem in *rebus ipſis* expoſui nuper (u): Idem etiam ſolvi poſſet in numeris, quia hic *quatuor* ſunt *partes* 1, 2, 3, 4, quarum *ſumma* eſt 10, eligendo ex Variationum Claſſe ^{10}D (v) eas tantummodo transpoſitiones, quae figuris conſtant 1, 2, 3, 4, excluſis reliquis, in quibus figurae occurrunt 5, 6, 7. Sed praeſtat, Problema quomodo ſolvatur ab huiusmodi Claſſibus independenter, oſtendiſſe.

6. Ubi rerum permutandarum aliquae ſunt eaedem, ut in *a, a, a, a, b, b, b*; poſitis pro literis iisdem, etiam Elementis numericis iisdem 1, 1, 1, 1, 2, 2, 2, permutationes itidem producuntur a numeris; *minimo* 1111222, *maximo* 2221111 et *intermediis* reliquis, a propoſitis et inter ſe diverſis, omnibus, in quibus 1 occurrit *quater*, 2 *ter*.

7. Similiter pro $\begin{matrix} a^4 & b^3 \\ 1 & 2 \end{matrix}$, cum ſit $4+3=7$, et $4.1+3.2=10$, determinari poſſent Permutationes per Claſſem ^{10}G (w), eligendo Complexiones tantummodo eas, in quibus occurrunt figurae numericae 1 et 2. Sic literarum $\begin{matrix} a^5 & b^2 & c^4 \\ 1 & 2 & 3 \end{matrix}$ transpoſitiones exhiberentur, quia $5+2+4=11$, et $5.1+2.2+4.3=21$, per Claſſem ^{21}L, eligendo tantummodo eas transpoſitiones, in quibus Elementa occurrunt 1, 2, 3, non alia, nempe 1, *quinquies*; 2, *bis*; 3, *quater*; poſitis autem pro figuris 1, 2, 3 literis *a*, *b*, *c* reſpective.

Hinc patet ſolutionum ſimul diverſitas et coniunctio mutua; nam Complexiones et Variationes numeri dati ſive propoſiti *n*, latiſſime patent.

8. PROBLE-

(t) *Infinit. Dignit.* §. *XXVII*, 8. *p.* 134.

(u) *In libello modo laud. not. s. p.* 93.

(v) *Infinit. Dignit. Append. Tab. X. p.* 177.

(w) *Infinit. Dign. Append. Tab. X. p.* 180.

8. PROBLEMA II. *Rerum datarum a, b, c, d, e, f . . . . reperire Complexiones, admiſſis repetitionibus, poſſibiles omnes rite ordinatas.*

SOLUTIO. Poſitis
pro *literis a, b, c, d, e, f . . . . ex ordine,*
figuris numericis 1, 2, 3, 4, 5, 6 . . . . *reſpective;*
Numeri ex ſingulis *binis*, *ternis*, *quaternis* etc *Elementis* vel *figuris*, ſive diverſis omnibus, ſive non diverſis, compoſiti hac lege, ut figuram praecedentem quamlibet vel *eadem*, vel *maior*, nunquam tamen minor, a *dextra* ſequatur, *ordine* ſcribantur omnes. Numeri inde oriundi ſiſtent omnes Complexiones poſſibiles, admiſſis etiam repetitionibus, rite ordinatas.

DEMONSTRATIO. Si negas: eſto v. c. ſi fieri poteſt, *aacdddf* Con7natio rite ordinata, cui non reſpondeat numerus, ex quo derivari potuiſſet. Ergo numerus 1134446 ex Elementorum, quibus conſtat, Con7natione produci non poteſt Q. E. A. Ergo Complexiones quaeſitae omnes exhibentur per huiusmodi numeros. Q. E. F.

9. EXEMPLUM. *Rerum trium a, b, c determinantur per numeros, ex figuris tantummodo 1, 2, 3, conſtantes, Complexiones, admiſſis repetitionibus, rite ordinatae, ſequentes:*

Uniones 1, 2, 3; et *Com2nationes* 11, 12, 13; 22, 23; 33; *et Con3nationes* 111, 112, 113, 122, 123, 133; 222, 223, 233; 333; cum reliquis, quae ſequuntur, *Con4nationibus*, *Con5nationibus*, etc.

10. Modus celerrime reperiundi *formas* quaeſitas omnes, conſiſtit in eo, ut *primo* ponantur *numeri* ipſi 1, 2, 3, 4, 5, 6, 7, 8, 9 . . . . tanquam *uniones*; *ſecundo*, ſingulis, inde ab 1, praeponatur 1, et poſtea ſingulis, inde a 2, praeponatur 2, poſtea ſingulis inde a 3, praeponatur 3, et ſic porro, usque ad 99 --- pro *Com2nationibus; tertio* ſingulis Com2nationibus praefigatur rurſus 1, deinde ſingulis, inde ab his qui incipiunt a 2, praefigatur 2, et, inde ab his qui incipiunt a 3, praefigatur 3, et ſic porro, usque ad 999 --- pro *Con3nationibus;* et ſic deinceps, poſitis Con3nationibus, pro *Con4nationibus;* et poſitis Con4nationibus, pro *Con5nationibus;* etc, quousque tandem progredi velimus.

11. Quod ſi pro numeris his adhibeantur literae, *a, b, c, d, e, f, g, h, i . . . .* poterunt ſtatim Complexiones in ipſis *rebus*, ſive *literis* exhiberi. Huc pertinet Tabula noſtra *ima* (x) qua exhibentur *Rerum datarum quocunque numero Complexiones ſecundum Exponentem, ſive Indicem quemlibet indefinitum n;* ſic, ut Complexionibus minorum Exponentium $n-1$, $n-2$, $n-3$ etc, non opus ſit, pro Complexionibus maximi Exponentis n determinandis.

(x) *Infinit. Dign. p. 157-159.*

Directrix harum *Complexionum*, h. e. formula cuius evolutione ſingulae Complexiones eo, quo debent, ordine, producantur, ponitur ibi

$$\overset{1}{a^n},\ \overset{2}{a^{n-1}B},\ \overset{3}{a^{n-2}B},\ \overset{4}{a^{n-3}B},\ a^{n-4}B\ \ldots\ldots\ \overset{m}{a^{n-m}B}.$$

12. Quod ſi non admittantur res repetitae, omittendi ſunt numeri, in quibus figurae eaedem repetuntur. Hoc fit, numeros hosce vel eligendo ex numeris in 10; vel praefigendo ibi ſingulas figuras numericas 1, 2, 3, 4, 5, 6, 7, 8 .... ſuo ordine Complexionibus, quae a 2, 3, 4, 5, 6, 7, 8, 9 .... incipiunt, reſpective, h. e. *minores* ſemper *maioribus*, non iisdem, aut minoribus. Complexiones hinc derivantur omnes eadem facilitate, quam in 10, ſed pauciores utique et numero finitae, pro rerum numero finito; cum reliquae ſint Complexiones numero infinitae, pro rerum etiam numero finito [II, 19.]

Huc pertinet Tabula noſtra IIda (y) *Complexionum rerum quotlibet datarum, in quibus nulla res ſive litera bis debet aut ſaepius occurrere*; cuius ibi poſuimus *formulam directricem*:

$$\overset{1}{A},\ \overset{2}{A},\ \overset{3}{A},\ \overset{4}{A},\ \overset{5}{A},\ \overset{6}{A},\ \overset{7}{A}\ldots\ldots\ \overset{m}{A}$$

13. EXEMPLUM. *Rerum datarum quinque a, b, c, d, e, determinantur, omiſſis repetitionibus, Complexiones rite ordinatae:*

Unionibus: a; b; c; d; e;
Com2nationibus: ab, ac, ad, ae; bc, bd, be; cd, ce; de;
Con3nationibus: abc, abd, abe, acd, ace, ade; bcd, bce, bde; cde;
Con4nationibus: abcd, abce, abde, acde; bcde;
Con5natione una: abcde.

Complexiones apparent hic in ipſis literis, pro quibus etiam potuiſſent poni figurae numericae 1, 2, 3, 4, 5.

14. *Complexiones numeri propoſiti n*, ſive definiti cuiuslibet, vel eliguntur ex Complexionibus ſimpliciter [10], quae numerum, addendo ſingulas figuras, efficiunt n; vel determinantur independenter eo modo facillime, quem alibi (z) oſtendi et demonſtravi.

15. PROBLEMA III. *Rerum a, b, c, d, e, f, g, h, i, k ... quocunque numero, quaerendae ſunt, admiſſis repetitionibus, Variationes poſſibiles omnes.*

SOLU-

(y) *Infin. Dign. p. 161.*

(z) *Infinit. Dignit. §. XXII. p. 74. ſeq.*

SOLUTIO I. Quaerantur *Complexiones* omnes [per 10], *Complexionumque* ſingularum omnes *Permutationes* [ex 3].

Duas itaque hic regulas, pro Complexionibus unam, pro Permutationibus alteram, adhibendo, ſatisfieri Problemati poteſt; ſed poteſt etiam, et longe commodius, unam tantummodo regulam ſequendo hanc:

SOLUTIO II*da*. Poſitis literis

1) *a, b, c, d, e, f, g, h, i, k* . . . . pro *Inionibus* in *Claſſe Ima;*
praefigantur literae hae ſingulae ex ordine:
2) *cunctis Inionibus,* pro *Com2nationibus* in *Claſſe 2da;* et
3) - - *Com2nationibus,* - *Con3nationibus* in *Claſſe 3tia;* et
4) - - *Con3nationibus,* - *Con4nationibus* in *Claſſe 4ta;* et
5) - - *Con4nationibus,* - *Con5nationibus* in *Claſſe 5ta;* et
ſic porro, quousque progredi velimus. Species inde oriundae ſiſtent rerum datarum quocunque numero Variationes omnes quaeſitas.

DEMONSTRATIO *Solutionis Imae* patet per ſe. In *Solutione IIda* oriuntur Com2nationes ab Inionibus, Con3nationes a Com2nationibus; etc. verbo, Conmnationes a Con($m-1$)nationibus. Haberi autem in Claſſe 2*da* Com2nationes hoc modo poſſibiles omnes, facile conceditur. Quod ſi itaque quis neget, adeſſe Conmnationem aliquam quamcunque, ... $pqpxf$ in Claſſe m, idem regrediendo etiam neget neceſſe eſt, adeſſe Con($m-1$)nationem, Con($m-2$)nationem . . . Con3nationem pxf, et Com2nationem xf, unde ortum trahere potuiſſet Conmnatio . . . . . $pqpxf$; quod eſt falſum: cum ſecunda Claſſis omnes habeat Com2nationes ὁμολογουμένως. Ergo Variationes exhibentur hoc modo poſſibiles omnes. Q. E. F.

16. Solutio Problematis tertii altera, qua datae res ſive literae, omnibus, quibus poſſunt, modis coordinantur et permutantur, coincidit cum ea, qua Polynomii $a+b+c+d+e+f+$ etc quaeruntur *Dignitates: ſecunda, tertia, quarta* etc per *multiplicationem.* Nam *Quadratum* v. c. reperitur, ſingulas radicis partes a, b, c etc. in ſingulas multiplicando, et *Cubus* producitur factis, a ſingulis radicis partibus in ſingulos Quadrati terminos; et ſic porro pro reliquis potentiis altioribus. Hinc peculiaris illa συμπάθεια inter *Complicationes* iſtas et *poteſtates polynomiorum*, quam iam notavit BERNOULLIUS (*).

17. Quomodo igitur res ipſae datae ſive literae variari poſſint, patet [ex 15]. Idem fit ponendo,

 pro

(*) *Arte Conject. P. II. C. VIII. p. 131.*

pro *literis a, b, c, d, e, f, g, h, i, ex ordine,*
Elementa numerica 1, 2, 3, 4, 5, 6, 7, 8, 9, *reſpective,*

Sed regula tunc, quam ſequi in configurandis Variationibus ex ordine oporteat, concipi breviſſime et proponi tribus verbis poteſt ſic: *Scribe numeros ex ordine, inde ab 1, tot, quot Variationes proferre propoſitum eſt, omiſſis his in quibus Cyphra ſive* 0 *occurrit, tanquam inutilibus.* Nam notae 0 nulla hic reſpondet res ſive litera.

Hinc *Iniones:* 1, 2, 3, 4, 5, 6, 7, 8, 9;
et *Com2nationes:* 11, 12... 19, 21... 29, 31... 39, 41 — 98, 99;
et *Con3nationes:* 111, 112... 119, 121... 129, 131 — — 998, 999;
et *Con4nationes:* 1111, 1112... 1119, 1121 — 1199, 1211 — — 9999;
et *Con5nationes:* 11111 — 99999; et *Con6nationes:* 111111 — 999999;
cum reliquis Complexionibus numero infinitis.

18. Vbi poſita fuiſſent,

pro *literis a, b, c, d, e, f, g, h, i, k ex ordine,*
Elementa 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 *reſpective,*

Cyphra hic, ſive 0, poſitum pro *a*, referretur inter *figuras ſignificantes:*

Hinc *Iniones:* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;
et *Com2nationes:* 00, 01, 02... 09, 10, 11... 20, 21 — 90, 91... 99;
et *Con3nationes:* 000, 001, 002 — 099, 100, 101 — 900, 901... 999;
et *Con4nationes:* 0000, 0001... 0009, 0010 — 9000, 9001... 9999;
etc. etc. etc. etc.

Scribantur nimirum pro

Inionibus: numeri omnes inde a 0, 1, 2, 3 ... *usque ad* 9;
Com2nationibus: — — — — 99;
Con3nationibus: — — — — 999;
Con4nationibus: — — — — 9999;
etc. etc. etc. etc.

adiunctis, ubi fieri poteſt, ſingulis Variationum Speciebus, *Cyphris* ad laevam tot, quot opus eſt, ut *Com2nationes* omnes *duabus*, *Con3nationes tribus*, *Con4nationes quatuor* etc *figuris* ſcriptae appareant.

Poni etiam pro 0 *res ipſa* poteſt *a*, reliquis numerorum ſignis manentibus. Quod ſi fit, adhibentur a^1 cum potentiis a^2, a^3, a^4 ... veluti *Complementa Dimenſionum* in ſingulis, ubi opus eſt, Speciebus Variationum. Maior inde adhuc utilitas redundat pro *Complexionibus* quaerendis rite ordinatis; et proinde adhibetur ſaepe in Logiſtica noſtra Serierum.

19. Quodſi

19. Quodſi *pauciores ſunt res* [quam in 17.] negligendi etiam ſunt numeri, tanquam *inutiles*, qui figuras habent, quibus res, ſive litera non reſpondet.

Sic poſitis pro *tribus* rebus a, b, c, *Elementis numericis* 1, 2, 3; *inutiles* cenſentur numeri, in quibus reperiuntur figurae reliquae, 4, 5, 6, 7, 8, 9.

Hinc reſultant [ex 17 vel 15, II]

Iniones: 1, 2, 3;

et *Biniones:* 11, 12, 13; 21, 22, 23; 31, 32, 33;

et *Terniones:*

111,	112,	113,	121,	122,	123,	131,	132,	133;
211,	212,	213,	221,	222,	223,	231,	232,	233;
311,	312,	313,	321,	322,	323,	331,	332,	333;

et reliquae reliquarum Claſſium *Variationes*, praefigendo nimirum pro Claſſe qualibet *ſequente* ſingulas figuras ſingulis ſpeciebus Claſſis *antecedentis*.

20. Quod ſi autem *numerus* rerum *maior* foret *denario*, ipſi numeri 10, 11, 12... pro *Elementis*, ſive *ſignis monadicis* uſurpari poſſunt in *Complicationibus* aeque ac *Variationibus*; et redit res ad *Syſtema plurium quam decem Elementorum ſive figurarum*. (β).. Solutio tamen propoſita [in 15] ſemper eadem manet; adhibitis tantum *commatibus*, ne confundantur 10, 11, 12... cum 1, 0; 1, 1; 1, 2 (γ).

21. Patet autem, *Numerum Variationum*, admiſſis repetitionibus, fieri adeo magnum, ut exhiberi omnes non poſſint. Idem fit pro Complexionibus. Sed maximo commodo Analyſis in problematum ſolutione, *initia* praeſcribit et *progreſſus Legem*, ſic, ut non omnes deſiderentur ſimul.

22. *Minuitur* etiam *Numerus Variationum* vehementer, ubi rebus repetitis opus non eſt, neglectis nimirum numeris, in quibus una figura plus vice ſimplici occurrit; aut ubi *Variationes* poſtulantur *numeri propoſiti n* ſive *definiti* cuiuslibet; quae, quomodo reperiantur numerorum ope (namque aliter vix poſſunt) e Variationibus ſimpliciter, ſive indefinitis, independenter, alibi (δ) oſtendimus.

b) *Numeri Permutationum, Complexionum ac Variationum.*

23. *Numerus Permutationum* rerum *n*, veluti $a^{\alpha}\, b^{\beta}\, c^{\gamma}\, d^{\delta}\, e^{\varepsilon} \ldots\ldots$ poſita particularium Exponentium ſumma $\alpha+\beta+\gamma+\delta+\varepsilon+ \text{etc} = n$ (ſive res ſint

(β) Conf. quae diximus de variis Syſtematibus in libelli (*not. s*) p. 97-100.

(γ) Exempla praebet *Infin. Dign. App. Tabula* IX, *p.* 173 *ſſ.* numerorum ex ſyſtemate 20 figurarum.

(δ) *Infin. Dignit.* §. *XXVII*, *ſ.* 4. *p.* 129. *ſeq.*

ſint omnes diverſae *a*, *b*, *c*, *d*, *e* ...... ubi nempe $\alpha=\beta=\gamma=\delta=\varepsilon=\text{etc}=1$; ſive aliquot earum recurrant) definitur hac formula (ε)

$$\frac{1.\ 2.\ 3.\ 4.\ 5.\ 6 \ldots\ldots n-3.\ n-2.\ n-1.\ n}{1.2\ldots\alpha\times 1.2\ldots\beta\times 1.2\ldots\gamma\times 1.2\ldots\delta\times 1.2\ldots\varepsilon\times 1.2.\ \text{etc.}}$$

vel, quae coincidit cum ea, ſed brevior paulo eſt, praeſertim ubi α magnus eſt numerus

$$\frac{n.\ n-1.\ n-2.\ n-3.\ n-4.\ n-5 \ldots\ldots (\alpha+1)}{1.\ 2.\ 3\ldots\beta\times 1.\ 2.\ 3\ldots\gamma\times 1.\ 2.\ 3\ldots\delta\times 1.\ 2.\ 3\ldots\varepsilon\times 1.\ 2.\ 3\ \text{etc}}$$

Poteſt etiam, loco α, quilibet alius Exponens β, γ, δ, ε ... *poni in Numeratore et omitti in Denominatore.* Sed refert *maximum* ex his eligere.

24. a) *Numerus Complexionum Exponentis m* a rebus datis diverſis *a*, *b*, *c*, *d*, *e* ..., multitudine *n*, *admiſſis repetitionibus*, exhibetur formula (ζ)

$$\frac{n.\ n+1.\ n+2.\ n+3.\ n+4.\ n+5 \ldots\ldots (n+m-1)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ \ldots\ldots m}$$

ſive, quae cum ea coincidit:

$$\frac{m+1.\ m+2.\ m+3.\ m+4.\ m+5 \ldots\ldots (m+n-1)}{1.\ 2.\ 3.\ 4.\ 5.\ \ldots\ldots (n-1)}$$

Hinc *Numerus omnium Complexionum*, ab *Inionibus* inde usque ad *mtiones* incluſive, producitur:

$$\frac{n+1.\ n+2.\ n+3.\ n+4 \ldots (n+m)}{1.\ 2.\ 3.\ 4 \ldots\ldots m} - 1.$$

b) *Numerus Complexionum* a rebus *a*, *b*, *c*, *d* ... diverſis *n*, *ubi res repetitae non admittuntur*, reperitur

pro *Inionibus* $= n$; *Com2nationibus* $= n.\ \frac{n-1}{1.\ 2.}$

pro *Con3nationibus* $= \frac{n.\ n-1.\ n-2}{1.\ 2.\ 3}$; *Con4nationibus* $= \frac{n.\ n-1.\ n-2.\ n-3}{1.\ 2.\ 3.\ 4.}$

adeoque pro *Combinationibus Exponentis*, ſive *Indicis m*, h. e. pro

Conmnationibus $= \frac{n.\ n-1.\ n-2.\ n-3 \ldots\ldots (n-m+1)}{1.\ 2.\ 3.\ 4 \ldots\ldots m}$

Hinc *Complexionum* per *omnes ſimul Claſſes Numerus* $= 2^n - 1$.

c) *Nu-*

(ε) *Infinit. Dignit.* §. XIII, *ſ.* 3. *p.* 32. Alia formulae forma exhibetur p. 31.
(ζ) *Infin. Dignit.* §. X, *ſ.* 10, *p.* 24, 25.

c) *Numerus Complexionum numeri propoſiti n* ſeu definiti cuiuslibet, (ſive res repetitae admittantur, ſive non admittantur) a tabula (*) pendet, eo modo, quem deſcripſimus.

25 α) *Numerus Variationum* rerum diverſarum *n*, *admiſſis repetitionibus*, in *1ma* Claſſe eſt *n*, in *2da* n^2, in *3tia* n^3, in *4ta* n^4 .... in *mta*, ſive pro *Variationis Exponente m*, eſt n^m; adeoque pro Claſſibus, inde a *1ma* usque ad *mtam*, incluſive, producitur *Variationum Numerus*

$$n^1+n^2+n^3+n^4+n^5+n^6 \ldots\ldots + n^m = \frac{n\,(n^m-1)}{n-1}$$

β) *Numerus Variationum a Numero quolibet propoſito n*, in ſingulis Claſſibus aequatur *Coefficientibus Binomialibus* Exponentis $n-1$ ex ordine, adeoque in *1ma* Claſſe eſt 1, in *2da* eſt $\frac{n-1}{1}$, in *3tia* eſt $\frac{n-1.\ n-2}{1.\ 2}$; in *4ta* eſt $\frac{n-1.\ n-2.\ n-3}{1.\ 2.\ 3.}$; etc, et proinde pro *omnibus Claſſibus coniunctim*, *Variationum Numerus eſt* $1+\frac{n-1}{1}+\frac{n-1.\ n-2.}{1.\ 2}+\frac{n-1.\ n-2.\ n-3}{1.\ 2.\ 3}+\text{etc}=2^{n-1}$ unde formula haec et altera [in 24, b] quomodo pendeant a ſe invicem, ſimul perſpicitur.

c) *Uſus rei combinatoriae breviter indicatur.*

Quaeſtiones combinatoriae tam late patent in omni ſcientiarum et artium genere, ut potiores tantum commemorare, difficillimum videatur. Huc, inter alia multa, pertinent: partium, quae rem aliquam conſtituunt, iuſta et completa enumeratio, earum, ubi opus eſt, omnimoda compoſitio, aut compoſitarum transpoſitio; rerum, vel certa aliqua lege vel etiam fortuito evenientium viciſſitudines, numerusque viciſſitudinum; veriſimilium coniecturae; eventuum ſimilium aut diſſimilium ſpes; ars aeſtimandi; omne genus ludorum, et ſortitionum (*Lotterien u. d. g.*) et magna pars Arithmetices politicae; Analyſis univerſa, cum Arte characteriſtica; Geometria cum Trigonometria utraque; Matheſis applicata, et, quae hinc pendent, Artes mechanicae; Phyſica experimentalis et Chemia; Logica probabilium; Logicae pars, quae dicitur, inventiva: Capita de propoſitionibus, diviſionibus,

(*) *Infin. Dignit. Append. Tab.* VII, *p.* 170, 171.

bus, ſyllogismis inductionibus, etc, et quae ſunt reliqua, in quibus fere regnat Ars Combinatoria, cuiusque ad ſcientias applicandae perfectiſſima exempla et omnibus numeris abſoluta praebet Trigonometria, plana et ſphaerica, quae totae conſtituuntur Combinationibus. In uſibus Artis combinatoriae demonſtrandis multus iamiam fuit LEIBNITIUS et IAC. BERNOULLIUS, qui duabus libri partibus poſterioribus de his expoſuit, ſed, quo minus poſſet plura afferre et maiora, morte praeventus fuit; et nemo eſt, qui, cognita re et perſpecta penitius, non utatur maximo cum fructu. Ex pluribus unum afferre ſatis ſit *Summum* LAMBERTUM; *Geometra-Philoſophum*, dum viveret, *ſubtiliſſimum* et *ingenioſiſſimum*, qui combinatoriis, tanquam optimis inſtrumentis, uſus eſt ſaepiſſime in rerum philoſophicarum indagatione; quod nemini poterit obſcurum eſſe, qui opera eius duo philoſophica maiora et praeſtantiſſima *Organum novum* et *Architectonices initia* (θ) attente legit — lecta intellexit.

Sunt etiam, quibus nihil videatur primo intuitu cum combinationibus et permutationibus commune eſſe, et quae tamen utiliter eo poſſunt referri ac deduci. Huc pertinent Serierum a rebus ſingulis, vel aliis Seriebus, oriundae Geneſes et quae hinc pendent theoremata ac problemata, quae *Methodum ſimul combinatoriam perfectam* et *generalem* exhibent, a nemine hactenus tentatam. Reduximus autem Rei combinatoriae ad Analyſin applicandae, vel cum Analyſi coniungendae, ſummam, ad problemata duo analytico-combinatoria: *Complexionum alterum*, ab *una eademque Serie elementari p* derivandarum; *alterum Variationum*, quaerendo terminorum ſingulorum Variationes a *Seriebus pluribus elementaribus* p, q, r, $s\ldots$ diverſis, coniunctim: effecimusque, ut, quae quaerantur circa Series, vel a *Dignitate*

$$(\alpha z+\beta z^2+\gamma z^3+\delta z^4+\varepsilon z^5+\text{etc})^m=p^m,$$

vel *Producto*

$$(\alpha z+\beta z^2\ldots)(az+bz^2\ldots)(\mathfrak{a}z+\mathfrak{b}z^2\ldots)(\mathfrak{A}z+\mathfrak{B}z^2\ldots)\,etc=pqrs\ldots$$

pendeant et determinentur. Modus, quo fieri poſſit ac debeat, tradetur in Logiſtica Serierum per formulas analytico-combinatorias.

§. IV.

Logiſticae Serierum exponitur Conſpectus per ſingula Capita.

Conſtituimus ſingula circa Series Theoremata, Problemata, et Operationes potiores maximeque compoſitas, ſingulis Tabulis ſiſtere. Tabularum erunt ex ordine tituli ſequentes:

TABULA

(θ) *Neues Organon, T. I, II, Leipzig 1764. Anlage zur Architektonik; T. I, II, Bresl. 1771.*

TABULA I. *Serierum in Series Multiplicatio.*

Producti a duabus, tribus, quatuor .... quotlibet Seriebus,

vel, $a + b + c + d + e + etc.$

vel, $\alpha z^{\mu} + \beta z^{\mu+\delta} + \gamma z^{\mu+2\delta} + \delta z^{\mu+3\delta} + \varepsilon z^{\mu+4\delta} + etc.$

vel denique, $a z^{\alpha} + b z^{\beta} + c z^{\gamma} + d z^{\delta} + e z^{\varepsilon} + etc.$

Terminus generalis et *termini ſinguli* per datas ipſas res, h. e. Seriei terminos ſive terminorum Coefficientes ſimpliciſſime expreſſi.

TABULA II, *Serierum per Series Diviſio.*

Quotientis, $\dfrac{a z^{\mu} + b z^{\mu+\Delta} + c z^{\mu+2\Delta} + d z^{\mu+3\Delta} + etc}{\alpha z^{\nu} + \beta z^{\nu+\Delta} + \gamma z^{\nu+2\Delta} + \delta z^{\nu+3\Delta} + etc}$;

vel, $\dfrac{a + b + c + d + e + etc}{\alpha z^{\nu} + \beta z^{\nu+\Delta} + \gamma z^{\nu+2\Delta} + \delta z^{\nu+3\Delta} + etc}$;

vel $\dfrac{a + b + c + d + e + etc}{\alpha + \beta + \gamma + \delta + \varepsilon + etc}$; etc, etc,

Terminus generalis cum *ſingulis terminis* ſimpliciſſime expreſſis.

TABULA. III. *Serierum Dignitates et Radices.*

Dignitatis: $(\alpha z^{\mu} + \beta z^{\mu+\Delta} + \gamma z^{\mu+2\Delta} + \delta z^{\mu+3\Delta} + etc.)^{m}$

vel, $(a + b + c + d \ldots)^{m}$; vel, $(\alpha z^{a} + \beta z^{b} + \gamma z^{c} + \delta z^{d} + etc.)^{m}$

quicunque tandem *numerus* ſit Exponens *m*, *Terminus generalis* cum *terminis ſingulis* ſimpliciſſimis.

TABULA IV. *Serierum in Series ſubſtitutio; itemque de ſubſtitutione continua.*

Datis Seriebus:

$$a y^{\alpha} + b y^{\beta} + c y^{\gamma} + d y^{\delta} + e y^{\varepsilon} \ldots = p;$$
$$A z^{\mu} + B z^{\mu+\Delta} + C z^{\mu+2\Delta} + D z^{\mu+3\Delta} \ldots = y;$$

Terminos exhibere, a *ſubſtitutione* valoris ipſius *y* in Seriem *p*, *provenientes.*

Porro, *poſitis Seriebus pluribus:*

$$z = A y + B y^{2} + C y^{3} + D y^{4} + E y^{5} + etc.$$
$$y = a x + b x^{2} + c x^{3} + d x^{4} + e x^{5} + etc.$$
$$x = \mathfrak{A} w + \mathfrak{B} w^{2} + \mathfrak{C} w^{3} + \mathfrak{D} w^{4} + \mathfrak{E} w^{5} + etc.$$
$$w = \mathfrak{a} v + \mathfrak{b} v^{2} + \mathfrak{c} v^{3} + \mathfrak{d} v^{4} + \mathfrak{e} v^{5} + etc.$$
$$etc. \quad etc. \quad etc. \quad etc. \quad etc.$$

 expri-

exprimere z per w, ſive per v, etc.

Item ubi, *poſita Serie*

$$ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.} = p,$$

pro x ipſa ponitur Series p, et efficitur P; et pro x in P denuo Series p, et efficitur Q; et pro x in Q rurſus Series p, et efficitur R; etc: *exhibere Terminos Serierum* P, Q, R, etc. *ſingulos*.

Denique ubi, *poſita Serie*

$$ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc} = p,$$

pro x ponitur Series $\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \text{etc.}$, et fit P; et pro x in P ponitur Series $\mathrm{A}x^1 + \mathrm{B}x^2 + \mathrm{C}x^3 + \text{etc.}$, et fit Q; et pro x in Q ponitur Series $\mathfrak{A}x + \mathfrak{B}x^2 + \text{etc.}$, et fit R; et ſic porro, Series aliae atque aliae novae ſubſtituuntur pro x, quae differunt inter ſe *Coefficientibus* variabilis x: *exprimere terminos Serierum* P, Q, R, etc. *ſingulos*.

TABULA V. *Incognitarum Eliminatio.*

Poſitis pluribus incognitis $z, y, x, w \ldots$ *totidemque aequationibus*

$$\begin{aligned}
\mathfrak{s}1 &= \mathfrak{z}1\, z + \mathfrak{y}1\, y + \mathfrak{x}1\, x + \mathfrak{w}1\, w + \text{etc.}\\
\mathfrak{s}2 &= \mathfrak{z}2\, z + \mathfrak{y}2\, y + \mathfrak{x}2\, x + \mathfrak{w}2\, w + \text{etc.}\\
\mathfrak{s}3 &= \mathfrak{z}3\, z + \mathfrak{y}3\, y + \mathfrak{x}3\, x + \mathfrak{w}3\, w + \text{etc.}\\
\mathfrak{s}4 &= \mathfrak{z}4\, z + \mathfrak{y}4\, y + \mathfrak{x}4\, x + \mathfrak{w}4\, w + \text{etc.}\\
\text{etc.} &\quad \text{etc.} \quad \text{etc.} \quad \text{etc.}
\end{aligned}$$

valorem determinare incognitarum $z, y, x, w \ldots$ per datos Coefficientes $\mathfrak{z}, \mathfrak{y}, \mathfrak{x}, \mathfrak{w} \ldots$. Solutio pendet a rerum datarum tot, quot ſunt incognitae ſive aequationes propoſitae, Permutationibus (*).

Porro, *datis aequationibus*

$$\text{P)}\; ay^m + by^{m-1} + cy^{m-2} + dy^{m-3} + \text{etc.} = 0$$

$$\text{Q)}\; \alpha y^n + \beta y^{n-1} + \gamma y^{n-2} + \delta y^{n-3} + \text{etc.} = 0$$

(ubi Coefficientes $a, b, c \ldots \alpha, \beta, \gamma \ldots$ ponuntur eſſe functiones rationales alius incognitae [x]) eliminare y, h. e. ex duabus aequationibus *indeterminatis* P et Q, facere *determinatam* R, ſolas variabilis x poteſtates complectentem. Hinc patebit ſimul: *Theorema Newtonianum* de relatione Coefficientium aequationis datae ad Summas poteſtatum ex aequationum radicibus (**), *Caſum* ſiſtere tantum *particularem* ex pluribus aliis numero infinitis, quorum aeque poterunt expeti et ſimul exhiberi Theoremata.

Hinc

(*) Pro *quatuor* igitur incognitis et *quatuor* Seriebus, a *Permutationibus* rerum a, b, c, d, exhibitis §. III, 4.

(**) *Arithmet. Univerſ.* p. 192. *Edit. Graveſ.*

Hinc tranſitus fit, ad plurium *Aequationum determinatarum* 𝔓, 𝔔, ℜ, 𝔖, 𝔗 .... ex pluribus *indeterminatis* P, Q; R, S, T ..., *plurium* incognitarum ſeu variabilium *functiones rationales* involuentibus, inventionem.

TABULA VI. *Irrationalium Sublatio.*

Seriem propoſitam ſive *Aequationem* in qua occurrunt *functiones irrationales*, ad *rationalem deducere*; tollenda Aſymmetria.

TABULA VII. *Serierum Transformatio.*

Seriem datam in aliam *transformare* propoſitae *conditionis*, vel *formae*: divergentem v. c. in convergentem, Summas in Producta, et contra; etc. etc.

TABULA VIII. *Serierum Interpolatio.*

Seriei y per poteſtates variabilis z progredientis, *determinare Coefficientes* ſic, ut

$$\text{poſito } z = a, b, c, d, e \ldots\ldots \overset{n-1}{a} \text{ ſucceſſive,}$$

$$\text{fiat } \quad y = \alpha, \beta, \gamma, \delta, \varepsilon \ldots\ldots \overset{n-1}{\alpha}, \text{ reſpective.}$$

Serierum huiusmodi ſiſti poſſunt *formae duae generales*, quibus Caſus quatuor a LAMBERTO ([λ]) exhibiti comprehenduntur cum reliquis numero infinitis.

TABULA IX. *Serierum Inverſio ſive Regreſſus.*

Propoſita aequatione

$$z^n = a y^\mu + b y^{\mu+\delta} + c y^{\mu+2\delta} + d y^{\mu+3\delta} + \text{etc.}$$

quaeritur Dignitas y^ϱ *expreſſa per* z.

Porro, *poſitis Seriebus*

$$\text{P)}\ a z^n + b z^{n+D} + c z^{n+2D} + d z^{n+3D} + e z^{n+4D} + \textit{etc.}$$

$$\text{Q)}\ \alpha y^\nu + \beta y^{\nu+\Delta} + \gamma y^{\nu+2\Delta} + \delta y^{\nu+3\Delta} + \varepsilon y^{\nu+4\Delta} + \textit{etc.}$$

aequalibus inter ſe, ſive P = Q, *exprimere* y *per* z, *ſive contra*, z *per* y; quicunque demum ſint numerorum n, ν, ϱ, D, Δ valores.

Item

(λ) *Zuſätze zu den logarithm. und trigon. Tabellen. Tab. XXXVIII. p. 198, 199.*

Item, poſito

$$z = \left\{ \begin{array}{l} a1\,y + a2\,y^2 + a3\,y^3 + a4\,y^4 + \ldots\ldots \\ + (b1\,y + b2\,y^2 + b3\,y^3 + b4\,y^4 + \ldots)\,z \\ + (c1\,y + c2\,y^2 + c3\,y^3 + c4\,y^4 + \ldots)\,z^2 \\ + (d1\,y + d2\,y^2 + d3\,y^3 + d4\,y^4 + \ldots)\,z^3 \\ \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \end{array} \right\}$$

Aut denique, $01 y^1 + 02 y^2 + 03 y^3 + 04 y^4 + \text{etc.}$

$$= \left\{ \begin{array}{l} (11 y^1 + 12 y^2 + 13 y^3 + 14 y^4 + \text{etc.})\,z \\ + (21 y^1 + 22 y^2 + 23 y^3 + 24 y^4 + \text{etc.})\,z^2 \\ + (31 y^1 + 32 y^2 + 33 y^3 + 34 y^4 + \text{etc.})\,z^3 \\ \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.} \end{array} \right\}$$

quaeritur z per y, aut y per z; e. q. s. r.

Hanc Tabulam totam iam abſolvi. Coefficientes terminorum hic ſubinde proveniunt valde compoſiti, quod aliter eſſe per rei naturam non poteſt, ſed Lex tamen, qua progrediuntur omnes per Elementa noſtra combinatoria fit manifeſta; quam, haud ſcio, an alia Methodus ulla exhibere poteſt. Caſum ſpecialem $z = ay + by^2 + cy^3 + dy^4 \ldots$ primus poſt NEWTONUM (μ) tractavit MOIVRAEUS (ν), et Legem progreſſus pronunciavit (ξ) ſed non demonſtravit, expoſitis ſimul terminis aliquot prioribus complexis. Idem tentavit etiam nuper pro pluribus terminis TEMPELHOFIUS (o), rerum analyticarum peritiſſimus artifex; ſed tamen nondum patet progreſſus Lex, quae continetur, poſito

$$y = \dot{\mathfrak{A}}z + \dot{\mathfrak{B}}z^2 + \dot{\mathfrak{C}}z^3 + \dot{\mathfrak{D}}z^4 + \text{etc.}$$

his formulis analytico-combinatoriis pro Coefficientibus fictis:

$$\dot{\mathfrak{A}} = \frac{1}{a};\quad \dot{\mathfrak{B}} = -\frac{b\,\mathfrak{b}^2 B}{a};\quad \dot{\mathfrak{C}} = -\frac{b\,\mathfrak{b}^3 B + c\,\mathfrak{c}^3 C}{a};$$

$$\dot{\mathfrak{D}} = -\frac{b\,\mathfrak{b}^4 B + c\,\mathfrak{c}^4 C + d\,\mathfrak{b}^4 D}{a};\quad \dot{\mathfrak{E}} = -\ \text{etc.}$$

et

(μ) *Epiſt. ad Oldenburg. poſterior cum Leibnitio communicanda. Leibn. Opp. T. III p. 75.*

(ν) *A Method of Extracting the Root of an Infinite Equation. Philoſ. Tranſact.* Vol. XX. p. 190.

(ξ) *Combine the Capital letters* (h. e. Coefficientes fictos vel aſſumtos) *as often, as you can make the Sum of their, Equal tho the Index of the Power to which they belong.*

(o) Subſtitutionibus ſeptem paginarum; *in den Anfangsgründen der Anal. endl. Gröſſen* p. 611-617.

et ſic porro, pro reliquis Coefficientibus, manifeſta progreſſus lege; deſignantibus nimirum hic Elementorum combinatoriorum $\mathfrak{b}^n\mathrm{B}$, $\mathfrak{c}^n\mathrm{C}$, $\mathfrak{d}^n\mathrm{D}$ .... (π)

numeris 1, 2, 3, 4, 5 .... *ex ordine*

Coefficientes fictos $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ .... *reſpective,*

quorum, quum ſit datus primus $\dot{\mathfrak{A}} = \frac{1}{a}$, dantur inde etiam reliqui $\dot{\mathfrak{B}}, \dot{\mathfrak{C}}, \dot{\mathfrak{D}}$ etc.

Sed alium reperi Coefficientes hos fictos determinandi modum faciliorem, per formulas:

$$\dot{\mathfrak{A}} = \frac{1}{a};\ \dot{\mathfrak{B}} = -\frac{\dot{\mathfrak{A}}b}{a^2};\ \dot{\mathfrak{C}} = -\frac{\dot{\mathfrak{A}}c + \dot{\mathfrak{B}}\mathfrak{b}^3\mathrm{B}}{a^3};$$

$$\dot{\mathfrak{D}} = -\frac{\dot{\mathfrak{A}}d + \dot{\mathfrak{B}}\mathfrak{b}^4\mathrm{B} + \dot{\mathfrak{C}}\mathfrak{c}^4\mathrm{C}}{a^4};\ \dot{\mathfrak{E}} = -\frac{\dot{\mathfrak{A}}e + \dot{\mathfrak{B}}\mathfrak{b}^5\mathrm{B} + \dot{\mathfrak{C}}\mathfrak{c}^5\mathrm{C} + \dot{\mathfrak{D}}\mathfrak{d}^5\mathrm{D}}{a^5};\ \text{etc.}$$

deſignantibus hic, Elementorum combinatoriorum $\mathfrak{b}^n\mathrm{B}$, $\mathfrak{c}^n\mathrm{C}$, $\mathfrak{d}^n\mathrm{D}$....

numeris 1, 2, 3, 4 .... *ex ordine*

literas *a*, *b*, *c*, *d* .... *reſpective;* (ρ)

Coefficientes ſcilicet *ſimplices*, non ipſas *Coefficientium Complexiones*, ad quas Analyſis vulgaris *Moivraeana* deducit, maximo incommodo. In Coefficientium ex his formulis evolutione optimum fuerit, ponere primo loco $a = 1$, et poſt redintegrare dimenſiones, per eiusdem factoris poteſtates a^1, a^2, a^3, a^4, ... quibus opus eſt.

Exhiberi etiam *ſtatim* poſſunt *Coefficientes* terminorum Seriei *y*, per *datos Coefficientes* *a*, *b*, *c*, *d* .... Sed non opus eſt his diutius immorarier.

TABULA X. *Varia circa Series Problemata.*

Functiones tranſcendentes: Sinus, Tangentes, Secantes per Arcus *polynomice* expreſſos, et contra, cum applicatione ad numeros; Serierum Logarithmi, et Logarithmorum Series; Arcus Parabolici, Elliptici, Hyperbolici; Series Brunkerianae; Numeri Bernoulliani; Reliqua.

Forte

(π) Explicantur haec Elementa *Infin. Dign. Tab.* IV, etiam V, poſitis ibi pro α, β, γ, δ... hic $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$...

(ρ) Poni itaque ſtatim poſſunt, *Infin. Dignit. Tabula Vta*, pag. 167, pro literis *ibi graecis* α, β, γ, δ, ε... *hic* literae *latinae*, h. e. *Coefficientes* a, b, c, d, e... reſpective.

Forte quaedam ex his peculiari Tabula proponentur. Accedent etiam, quae nondum attigi; namque in praeſentiarum exponere tantum volui de his, quae partim iam tentavi, partim vero etiam abſolvi. De Radicum ex Aequationibus inventione vel approximatione, quo Serierum ducunt Inverſiones; de Serierum Summis etc, non iam audeo pronunciare, quia de his nondum cogitavi, quantum dignitas materiae poſtulat. Sed veriſimile eſt, hanc noſtram notationem, quae per ipſa Elementa procedit, unde ortum trahunt Series, hunc etiam uſum nobis praeſtare poſſe.

Opus continebit Tabulas duplicis generis: *ſubſidiarias* alteras, quae *ſimpliciſſimas operationes combinatorias* exponent, literarum, vel numerorum ſubſidio; quales ſunt, quas in Appendice Methodi Potentiarum deſcripſi, decem numero (*); alteras *analyticas*, quas modo memoravi et recenſui, *operationes combinatorias magis compoſitas*, ſive formulas analytico-combinatorias complectentes. Utrasque dabimus, quoad poterimus, et emendatiſſimas, et uſibus, quibus ſunt deſtinatae, aptiſſimas.

De formularum huiusmodi natura, itemque formationis ac progreſſus lege, diligenter iamiam expoſitum eſt in libello de Infinitinomii Dignitatibus, et pluribus etiam dicetur in *Artis Combinatoriae Novo Syſtemate*. Ponentur autem formulae cum explicatione, quantum ſatis erit ad intelligentiam et uſum, relatis etiam breviter doctiſſimorum Virorum, NEWTONI, LEIBNITII, MOIVRAEI, BERNOULLIORUM, COLSONI, EULERI, LA GRANGII, KAESTNERI, LAMBERTI, SEGNERI, et aliorum, meditationibus, ſuo loco, interdum etiam formulis. In explicatione ipſa quomodo verſabimur, intelligi quodammodo poterit ex his, quae Tabulis duabus prioribus, quae ſpeciminis loco in calce libri propoſitae ſunt, illuſtrandis afferemus, ubi prius de Serierum ipſarum Serierumque terminorum ſingulorum ac coefficientium *ſignis* et *notationibus*, quibus utemur in formulis, ſignorumque inter ſe *relationibus* expoſuerimus.

§. V.

Signorum Explicationes.

Signa, quibus utimur, pleraque quidem nova ſunt, ſed tamen ad aliorum analogiam efficta. Operam autem dedimus, quandoquidem bona characteriſtica maximum eſt meditandi ſubſidium, ut notiones ſubiectas, quan-

(*) *Infin. Dignit. Append. inde a pag.* 153-180.

quantumvis etiam compoſitas, facillime animo repraeſentent; eorum maxime ex genere, quae Franco-Galli *ſignes parlans* nominant, et quorum, qui comites plerisque adduntur, numeri efficiunt, ut pro ſcientificis haberi et tractari poſſint.

A) *Serierum et terminorum ex Seriebus ſigla.*

1. Literae p, q, r, s, t, u, v, w, x, y, z interdum etiam π, ρ, σ, τ etc, brevitatis cauſa, pro *Seriebus* ipſis datis ponuntur v. c.

$$a + b + c + d + e + f + g + h + etc = \pi$$

$$\alpha y^{m} + \beta y^{m+d} + \gamma y^{m+2d} + \delta y^{m+3d} + \epsilon y^{m+4d} + etc = z$$

Hic Series π, in qua Polynomii partes indiscriminatim progrediuntur, ſic, ut communem aliquam quantitatem, ſecundum quam ordinari poſſent termini, non habeant, ab altera z, quae per quantitatis indeterminatae y poteſtates ſucceſſivas Exponentium m, $m + d$, $m + 2d$ *etc.* progreditur, vel ipſa notatione (π et z), ſi opus eſt, diſtingui poteſt. Series omnes ad duas haſce Claſſes referri poſſunt (r).

2. Literae t, vel ⌉ Serierum *terminum quemlibet* integrum cum Coefficiente, literae autem c, vel $\varkappa$ ſolum *termini Coefficientem*, in univerſum deſignant, adſcriptis tamen, ſeriei, vel ſerierum, ad quas pertinet terminus ſeu Coefficiens, ſigno ſive ſignis *a laeva*, membri autem inde oriundi, quod volumus indicare, numero, vel definito 1, 2, 3, 4 etc, vel indefinito m, n, μ, ν, ... *a dextra.*

Hinc $\pi\rho$t4 ſignat terminum quartum Producti a Seriebus π et ρ; et $pqrs$ ...⌉m, terminum $m$tum Producti a Seriebus pluribus p, q, r, s ...., quarum multitudo non determinatur.

$\frac{r}{s}\varkappa(n+1)$ indicat Coefficientem termini $(n+1)$ti, quotientis $\frac{r}{s}$, h. e. Seriei r diviſae per Seriem s; et $y^{\nu-1}$ $\varkappa(m+n)$, coefficientem $(m+n)$ tum Dignitatis $y^{\nu-1}$, nempe Seriei y ad dignitatem $(\nu-1)$ tam ſublatae.

Log. π^{n} c $(\mu+2)$ Coefficientem notat termini $(\mu+2)$ti, et Log. π^{n} t$(\mu+2)$ terminum ipſum integrum $(\mu+2)$ tum, ſeriei utrumque pro Logarithmo, pertinente ad Dignitatem π^{n}, ſeriei ſcilicet π, Exponente n affectae.

B) *Coefficientium notationes.*

In Serierum tractatione duo ſunt ſpectanda: *Exponentium* in terminorum quaeſitorum Dignitatibus *inventio*, et *Coefficientium* pro his terminis *determinatio;*

(r) *Infinit. Dignit.* §. III.

terminatio; illa Seriei *formam*, haec *materiam* conſtituit; illic NEWTONI Parallelogrammum analyticum, huc pertinet LEIBNITII *Methodus Serierum*, per ſuppoſitionem Seriei quaeſitae tanquam inventae. Amborum Methodi univerſales illae quidem, ſed difficultatibus non carent quibusdam in caſibus Quae tradentur hic de *Coefficientium* notatione, poterunt etiam, maximam partem, ad *Exponentes* transferri.

Sunt autem Coefficientes vel *dati*, vel *quaeſiti*, pro quibus etiam interdum *ficti* ponuntur, qui aſſumuntur tanquam dati. Dati hic intelliguntur *Polynomiorum*, adeoque latiſſime patentes, quandoquidem Polynomia etiam *Binomia* et terminos *monadicos* ſive *ſolitarios* comprehendunt, et maxime ſunt diverſi; quaeſiti ſunt vel *binomiales*, vel *polynomiales*, vel *geniti ex his* atque ortum trahentes. De ſingulis ſuo loco videbimus.

Notari autem poſſunt Coefficientes pluribus modis diverſis:

α) Sic, ut appareat, ad Series eos vel easdem, vel diverſas pertinere.

3. *Coefficientes dati* ſignentur in univerſum, more conſueto, literis Alphabeti cuiuslibet, A, B, C ... a, b, c, ... A, B, Γ .... α, β, γ ... 𝔄, 𝔅, ℭ ... 𝔞, 𝔟, 𝔠 ... latini, graeci, germanici etc, maioribus, minoribus, quae ipſa conſecutione ordinem Coefficientium deſignent.

Opus hic eſt interdum pluribus plurium Canonum literis pro Seriebus, quae haud raro in uno problemate proponuntur diverſis, v. c. ubi Claſſium poſtulantur Complexiones ſive Producti, a plurium ſerierum multiplicatione oriundi, termini.

4. *Coefficientes ficti*, ſive *aſſumti*, (quorum more *Leibnitiano*, per ſucceſſum, terminos homogeneos comparando, ſive etiam aliter, definiuntur valores ex datis) notentur itidem ſinguli Alphabeti cuiusque literis ſingulis, ſed punctis ſupra ſcriptis, veluti $\dot{A}$, $\dot{b}$, $\dot{\Gamma}$, $\dot{\delta}$, $\dot{ℭ}$, $\dot{𝔣}$ . . . .

Nempe Coefficientes fictos a datis *diſtinguere*, et *iisdem literis notare* ſimul, plurimum prodeſt, propter relationem mutuam, quae intercedit, et quia, quod quaeritur, a datis homonymis diſtingui, in tanta ſaepenumero Coefficientium, qui occurrere poſſunt, varietate, commodius vix queat.

β) Vel ut ſimul Serierum quidam ordo obſervetur.

5. In *ordine Serierum* definito obſervando poterunt Coefficientes *primus*, *ſecundus*, *tertius*, *quartus*, *quintus* .... *n tus*, (n+1) *tus* Seriei

primae, per $a1, a2, a3, a4, a5 \ldots\ldots an, a(n+1)$
ſecundae – $b1, b2, b3, b4, b5 \ldots\ldots bn, b(n+1)$
tertiae – $c1, c2, c3, c4, c5 \ldots\ldots cn, c(n+1)$
quartae – $d1, d2, d3, d4, d5 \ldots\ldots dn, d(n+1)$

communi

communi ſcilicet omnes literæ, $a, b, c, d \ldots\ldots$, adiuncto loci a dextra numero $1, 2, 3, 4, 5 \ldots\ldots n, n+1$, ſignari; ſimiliterque pro aliis etiam plurium ſerierum Coefficientibus datis, vel fictis (quibus tantummodo additur punctum (4) in vertice $\dot{a}1, \dot{a}2 \ldots \dot{b}3, \dot{b}4 \ldots \dot{c}5, \dot{c}6 \ldots \dot{d}n, \dot{d}\,(n+1)$ etc) per reliquas literas huius, vel alius cuiusque Alphabeti; qui mos apud optimos quosque Auctores iamdudum invaluit.

Interdum pro plurium ſerierum $p, q, r \ldots x, y, z$ (1) Coefficientibus ex ordine, ſoli numeri 1, 2, 3, 4, 5 .... utiliter ſcribi poſſunt, cum relatione ad ſeries, ad quas pertinent, hoc modo:

$$11x + 12x^2 + 13x^3 + 14x^4 + 15x^5 + \text{etc} = p$$
$$21x + 22x^2 + 23x^3 + 24x^4 + 25x^5 + \text{etc} = q$$
$$31x + 32x^2 + 33x^3 + 34x^4 + 35x^5 + \text{etc} = r$$

pro tribus Seriebus p, q, r, ubi prima nota Seriem, altera Seriei Coefficientem numerat, ſic ut v. c. 23 Seriei 2dae q Coefficientem 3tium deſignet, more Leibnitiano (*). Pro terminis in quibus x non apparet, ſcribitur 10, 20, 30

γ) Vel in relatione ad Serierum Variabiles diverſas, quarum ſunt Coefficientes.

6. Ubi *plures* occurrunt in *pluribus* ſeriebus *Variabiles*, convenit harum Coefficientes literis homonymis, ſed alius Alphabeti, diſtinguere, ſic, ut ſimul Ordinis Serierum (5) ratio habeatur; E. g.

$$S1 = \mathfrak{z}1\, z + \mathfrak{y}1\, y + \mathfrak{x}1\, x + \mathfrak{w}1\, w + \mathfrak{v}1\, v + etc$$
$$S2 = \mathfrak{z}2\, z + \mathfrak{y}2\, y + \mathfrak{x}2\, x + \mathfrak{w}2\, w + \mathfrak{v}2\, v + etc$$
$$S3 = \mathfrak{z}3\, z + \mathfrak{y}3\, y + \mathfrak{x}3\, x + \mathfrak{w}3\, w + \mathfrak{v}3\, v + etc$$
$$S4 = \mathfrak{z}4\, z + \mathfrak{y}4\, y + \mathfrak{x}4\, x + \mathfrak{w}4\, w + \mathfrak{v}4\, v + etc$$
$$S5 = \mathfrak{z}5\, z + \mathfrak{y}5\, y + \mathfrak{x}5\, x + \mathfrak{w}5\, w + \mathfrak{v}5\, v + etc$$
$$etc \qquad etc \qquad\qquad etc \qquad\qquad etc$$

Coefficientes ficti etiam hic notentur punctis, $\dot{\mathfrak{z}}1, \dot{\mathfrak{y}}2, \dot{\mathfrak{x}}3, \dot{\mathfrak{w}}4, \dot{\mathfrak{v}}5, \dot{\mathfrak{u}}\,(n+m)$

δ) vel in relatione ad Exponentes Variabilium, quarum ſunt Coefficientes.

7. Coefficientium ad *Exponentes Variabilium*, quarum ſunt Coefficientes, relationes deſignentur

α) Vel per Alphabeti literas quascunque; addito a dextra Variabilis Exponente n, ſed uncinis incluſo, hoc modo:

$c^{(n)}, C^{(n)}, \gamma^{(n)}, \Gamma^{(n)}, \mathfrak{c}^{(n)}, \mathfrak{C}^{(n)} \ldots$; ad evitandam ſcilicet numeri relativi (n) cum Exponente n aequivocationem in termino $c^{(n)} z^n$

 Hic

(*) *Opp. Leibn. T.* III, *p.* 416. *Conf. Eiusd. T. p.* 529, 530, 532, 533.

Hic $c^{(n)}$, $C^{(n)}$ . . . non determinat (ut cn, Cn . . . . in β) quotus ſit ordine Coefficiens $c^{(n)}$ vel $C^{(n)}$, ſed eum pertinere tanquam factorem, ad poteſtatem indeterminatam z^n; quo fit, ut pro diverſitate Exponentis n in z^n, poſſit etiam numerus relativus (n) varios induere valores, poſitivos et negativos, integros et fractos. Sic, v. c. poſita ſerie

$$\alpha z^m + \beta z^{m+d} + \gamma z^{m+2d} + \delta z^{m+3d} + etc$$

Coefficiens pertinens ad dignitatem z^{m+nd} deſignabitur hic generaliter per $c^{(m+nd)}$, quicunque demum ſint literarum m, n, et d numerici valores. Idem, ſed pro numero tantum integro n (ſecundum β) foret $c\,(m+nd)$ Caeterum literae c, C . . . (initiales ſcilicet vocabuli *Coefficiens*) adhibentur tantum hic exempli cauſa; nam et reliquae poſſunt ad hunc uſum, ubi opus eſt, literae depromi.

Pro Coefficientibus fictis ſcribitur hic $\bar{c}^{(n)}$, $\bar{C}^{(n)}$, $\bar{\gamma}^{(n)}$, $\dot{\Gamma}^{(n)}$, $\dot{c}^{(n)}$, $\bar{\mathfrak{C}}^{(n)}$ . . .

b) vel per literas homonymas quidem cum his, quibus Exponentes indicantur, ſed alius Alphabeti; v. c.

$$\alpha z^a + \beta z^b + \gamma z^c + \delta z^d + \varepsilon z^e + \text{etc.}$$

ubi Coefficientes $\alpha, \beta, \gamma, \delta, \varepsilon$ . . . . ab Exponentibus homonymis a, b, c, d, e . . . nullo modo ponuntur pendere, ſed ad eundem ſeriei terminum utrique pertinere, reſpective.

Pro Coefficientibus fictis ponitur hic ſimili modo

$$\mathfrak{A} z^a + \mathfrak{B} z^b + \mathfrak{C} z^c + \mathfrak{D} z^b + \mathfrak{E} z^e + \text{etc.}$$

c) ſive denique, ubi plurium Variabilum Producta in pluribus Seriebus occurrant, quarum comparatio inſtitui debeat, per Serierum numeros ordinales et Variabilium Exponentes ſimul

A) $1024\, x^0 y^2 z^4 + 1231\, x^2 y^3 z^1 + 1312\, x^3 y^1 z^2 + \text{etc}$
B) $2105\, x^1 y^0 z^5 + 2123\, x^1 y^2 z^3 + 2132\, x^1 y^3 z^2 + \text{etc}$
C) $3150\, x^1 y^5 z^0 + 3321\, x^3 y^2 z^1 + 3411\, x^4 y^1 z^1 + \text{etc}$

Ubi prior nota Seriem, reliquae terminos definitos per Exponentes definitos exprimunt (φ).

a) *Vel in relatione, quam habent Coefficientes inter ſe, ſecundum vel* a) *magnitudinem, vel* b) *diſtantiam.*

8. a) Coefficientium plurium inter ſe relationes, quoad *magnitudinem*, ſecundum quam, poſito aliquo Coefficiente (ſive ſimplex ille quantitas ſit, ſive complexa) pro Unitate, reliqui omnes Coefficientes ad hunc rationem habent,

(φ) *Leibn. Opp. T. III. p. 513, 514. Conf. not. praeced. u*

habent, ut numeri (integri vel fracti, poſitivi et negativi) ad Unitatem, optime et ſimpliciſſime per hos ipſos numeros exhibentur; quo fit, ut numeri, qui hic adhibentur, *ſuppoſititii* ſcilicet per hypotheſin, ſimul fiant *veri*, quorum in locum; retro deinde poſſint, Coefficientes dati, habita quantitatis pro Unitate poſitae reſpectu, facillime reſtitui.

b) Coefficientium relationes, quoad *diſtantiam*, quam habent inter ſe, exprimi poſſunt ſic, ut cum literis, quibus deſignantur, ſimul diſtantia ab uno aliquo Coefficiente, numeris ſupra ſcriptis, exponatur.

Sic Coefficientium ex ordine, inde a primo *a*, usque ad n^{tum} n

a, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, *l*, *m*, *n*, *o*, *p* . . . . . n . . . .
poterit

(1) *Indefinitus* quilibet *Coefficiens* per *indefinitum* n relative exprimi, ſupra ſcripto ſcilicet diſtantiae ipſius ab eo, quem deſignare velis, *antrorſum* vel *retrorſum* numero. Sic v. c.

$\overset{0}{n}$ Coefficientem ſignificat n^{tum} ex hypotheſi; $\overset{-1}{n}$, antecedentem proxime $(n-1)$ tum; $\overset{+1}{n}$, ſequentem proxime $(n+1)$ tum; ſimiliterque $\overset{-m}{n}$, vel $\overset{+m}{n}$ antecedentem, $(n-m)$ tum, vel ſequentem $(n+m)$ tum Coefficientem

Hinc $a = \overset{n-1}{n}$; $\overset{0}{a} = \overset{n-a}{n}$; $\overset{-1}{a} = \overset{n-m}{n} = \overset{-(m-1)}{n}$; $\overset{1-m}{b} = \overset{n+m}{n}$; $\overset{m+2}{c} = \overset{n+r}{n}$; $\overset{r+3}{}$ etc.

(2) *definitus* quilibet *Coefficiens* per *definitum* alium deſignari, notata huius ab illo diſtantia. Sic poſito $\overset{0}{g} = g$ fit:

$$\overset{-1}{g}=f;\quad \overset{-2}{g}=e;\quad \overset{-3}{g}=d;\quad \overset{-4}{g}=c;\quad \overset{-5}{g}=b;\quad \overset{-6}{g}=a;$$
$$\overset{+1}{g}=h;\quad \overset{+2}{g}=i;\quad \overset{+3}{g}=k;\quad \overset{+4}{g}=l;\quad \overset{+5}{g}=m;\quad \overset{+6}{g}=n;$$

et $\overset{-7}{g}=\overset{-8}{g}=\overset{-9}{g}=\overset{-10}{g}=etc.=0$, ubi nempe Series Coefficientium incipit cum *a*, atque ordine literarum recepto progreditur; ſimiliterque fit $\overset{+7}{g}=\overset{+8}{g}=\overset{+9}{g}=\overset{+10}{g}=etc.=0$, ubi deſinere ponitur cum litera *n*.

Exprimi itaque poſſent

$$\text{Coefficientes}\quad a,\ b,\ c,\ d,\ e,\ f,\ g\ \ldots\ldots\ \pi,\ \varrho,\ \sigma,\ \tau\ \ldots$$

$$\text{poſitis}\quad \overset{0}{a},\ \overset{1}{a},\ \overset{2}{a},\ \overset{3}{a},\ \overset{4}{a},\ \overset{5}{a},\ \overset{6}{a}\ \ldots\ \text{omnes per } a$$

$$\text{vel}\ -\quad \overset{-1}{b},\ \overset{0}{b},\ \overset{1}{b},\ \overset{2}{b},\ \overset{3}{b},\ \overset{4}{b},\ \overset{5}{b}\ \ldots\ -\ -\ b$$

$$-\ -\quad \overset{-2}{c},\ \overset{-1}{c},\ \overset{0}{c},\ \overset{1}{c},\ \overset{2}{c},\ \overset{3}{c},\ \overset{4}{c}\ \ldots\ -\ -\ c$$

$$-\ -\quad \overset{-3}{d},\ \overset{-2}{d},\ \overset{-1}{d},\ \overset{0}{d},\ \overset{1}{d},\ \overset{2}{d},\ \overset{3}{d}\ \ldots\ -\ -\ d$$

$$-\ -\quad \overset{-4}{e},\ \overset{-3}{e},\ \overset{-2}{e},\ \overset{-1}{e},\ \overset{0}{e},\ \overset{1}{e},\ \overset{2}{e}\ \ldots\ -\ -\ e$$

$$-\ -\quad \overset{-5}{f},\ \overset{-4}{f},\ \overset{-3}{f},\ \overset{-2}{f},\ \overset{-1}{f},\ \overset{0}{f},\ \overset{1}{f}\ \ldots\ -\ -\ f$$

$$-\ -\quad \overset{-6}{g},\ \overset{-5}{g},\ \overset{-4}{g},\ \overset{-3}{g},\ \overset{-2}{g},\ \overset{-1}{g},\ \overset{0}{g}\ \ldots\ -\ -\ g$$

etc etc etc etc.

$$\text{Hinc}\quad \overset{0}{g}=\overset{1}{f}=\overset{2}{e}=\overset{3}{d}=\overset{4}{c}=\overset{5}{b}=\overset{6}{a}$$

$$\text{et}\quad \overset{-1}{g}=\overset{0}{f}=\overset{1}{e}=\overset{2}{d}=\overset{3}{c}=\overset{4}{b}=\overset{5}{a}$$

et ſic porro pluribus aliis modis, qui a Schemate pendent omnes generali ſequente:

Literis ſive Coëfficientibus, $a, b, c, d \ldots \pi, \varrho, \sigma \ldots$ iungantur in vertice, numeri integri ex ordine n, $n-1$, $n-2$, $n-3$, ... $n-\mu$, $n-(\mu+1)$, $n-(\mu+2)$ ... diſtantiam exprimentes, qua quaelibet litera a *communi* aliquo omnium *termino* (litera nempe ea, cuius ponitur *diſtantiae Exponens* $=0$) removetur, et exhibebitur *ſcala relationis* huiusmodi:

n	$n-1$	$n-2$	$n-3$	$n-4$	$n-5$	$n-6$	$n-7$	$n-8$	$n-9$	$n-10$	$n-11$
$a,$	$b,$	$c,$	$d,$	$e,$	$f,$	$g,$	$h,$	$i,$	$k,$	$l,$	$m\ldots\ldots$

	$n-\mu$	$n-(\mu+1)$	$n-(\mu+2)$	$n-(\mu+3)$
......	$\pi,$	$\varrho,$	$\sigma,$	τ ....

Hinc patet:

a) Terminos propoſitae Seriei omnes, propter ipſum hunc terminum communem, inter ſe aequales eſſe, et proinde quam maxime diverſis modis exprimi Coefficientes poſſe.

$$\overset{n}{a}=\overset{n-1}{b}=\overset{n-2}{c}=\overset{n-3}{d}=\overset{n-4}{e}\ldots=\overset{n-\mu}{\pi}=\overset{n-(\mu+1)}{\varrho}=\overset{n-(\mu+2)}{\sigma}\ \ldots\ \text{etc.}$$

$$\text{et fore}\quad \overset{\mu}{a}=\pi;\quad \overset{\mu+1}{a}=\overset{\mu}{b}=\varrho;\quad \overset{\mu+2}{a}=\overset{\mu+1}{b}=\overset{\mu}{c}=\sigma\quad \text{etc.}$$

b) Numerum

b) Numerum quemlibet indefinitum n, $n-1$, $n-2$, etc, cuilibet numero definito integro, poſitivo et negativo, etiam Nihilo aequari atque hinc ſcalam relationis utcunque variari poſſe.

Sic. v. c. ubi quaeritur Scala, in qua ſit $\overset{-3}{g}$, ponatur $n-6=-3$; adeoque $n=3$, et ſit:

$$\overset{3}{a}, \overset{2}{b}, \overset{1}{c}, \overset{0}{d}, \overset{-1}{e}, \overset{-2}{f}, \overset{-3}{g}, \overset{-4}{h}, \overset{-5}{i}, \overset{-6}{k}, \overset{-7}{l}, \overset{-8}{m}, \overset{-9}{n} \ldots \overset{3-\mu}{\pi}, \overset{2-\mu}{\varrho} \ldots$$

unde prodeunt valores Coefficientis

$$\overset{-3}{g}\left\{\begin{array}{l} =\overset{-2}{f}=\overset{-1}{e}=\overset{0}{d}=\overset{1}{c}=\overset{2}{b}=\overset{3}{a}. \\ =\overset{-4}{h}=\overset{-5}{i}=\overset{-6}{k}=\overset{-7}{l}=\overset{-8}{m}=\overset{-9}{n}\ldots=\overset{3-\mu}{\pi}=\overset{2-\mu}{\varrho}\ldots\end{array}\right.$$

Similiter pro valore literae l, ponatur $\overset{0}{l}=\overset{n-10}{l}$, adeoque $n=10$, unde ſit

$$\overset{10}{a}, \overset{9}{b}, \overset{8}{c}, \overset{7}{d}, \overset{6}{e}, \overset{5}{f}, \overset{4}{g}, \overset{3}{h}, \overset{2}{i}, \overset{1}{k}, \overset{0}{l}, \overset{-1}{m}, \overset{-2}{n}, \overset{-3}{o}, \overset{-4}{p}, \overset{-5}{q}\ldots \overset{10-\mu}{\pi}, \overset{9-\mu}{\varrho}\ldots$$

$$\text{et } \overset{0}{l}\left\{\begin{array}{l} =\overset{1}{k}, \overset{2}{i}, \overset{3}{h}, \overset{4}{g}, \overset{5}{f}, \overset{6}{e}, \overset{7}{d}, \overset{8}{c}, \overset{9}{b}, \overset{10}{a}. \\ =\overset{-1}{m}, \overset{-2}{n}, \overset{-3}{o}, \overset{-4}{p}, \overset{-5}{q}, \overset{-6}{r}, \overset{-7}{s}, \overset{-8}{t}, \overset{-9}{u}, \overset{-10}{v}\ldots \overset{10-\mu}{\pi}, \overset{9-\mu}{\varrho}\ldots\end{array}\right.$$

quo ſignificatur, poſſe v. c.

Coefficientem $\overset{-3}{g}$ vel d; per $\overset{3}{a}, \overset{-2}{f}, \overset{1}{c}, \overset{-9}{n}, \overset{-4}{h}, \overset{2-\mu}{\varrho}, \overset{1-\mu}{\sigma}\ldots$

et — — — $\overset{0}{l}$ vel l, per $\overset{10}{a}, \overset{5}{f}, \overset{8}{c}, \overset{-2}{n}, \overset{3}{h}, \overset{-8}{t}, \overset{9-\mu}{\varrho}, \overset{8-\mu}{\sigma}\ldots$

aut quibuslibet aliis indicari, quibus commodum videbitur.

Eadem notatio locum etiam habet in Coefficientibus fictis $\dot{a}, \dot{b}, \dot{c}, \dot{d}$ etc. et pariter ſit $\dot{g}=\overset{6}{\dot{a}}=\overset{4}{\dot{c}}=$ etc; $\overset{n-1}{\dot{a}}=\overset{0}{n}$; $\overset{n+m}{\dot{b}}=\overset{n+m-1}{c}$; etc.

Cavendum tamen, ne *diſtantiae Exponentes*, quales hic adhibuimus, confundantur cum *ordinis Exponentibus*, qui pariter ſolent (numerorum certe figuris in Arithmetica) ſupra ſcribi, et quorum ſubſidio, ut alibi iamiam monui, in ipſa etiam Analyſi, Serierum poſſunt

$$\alpha z^{n}+\beta z^{n-1}+\gamma z^{n-2}+\delta z^{n-3}+\varepsilon z^{n-4}$$

termini integri per ſolos Coefficientes breviter notari hoc modo:

$$\overset{n}{\alpha}+\overset{n-1}{\beta}+\overset{n-2}{\gamma}+\overset{n-3}{\delta}+\overset{n-4}{\varepsilon}+\text{etc.}$$

ad analogiam ſcilicet *Algorithmi decadici, dyadici, ſexageſimalis*, aliorumque cuiuscunque ſcalae Syſtematum, in quibus figurarum (h. e. Coefficientium

tium $\alpha = c^{(n)}$; $\beta = c^{(n-1)}$; etc. [7]) ordines, ſive ordinum Indices o, ± 1, ± 2, ± 3 ..., $\pm (n-1)$, $\pm n$, notari ſolent numeris homonymis ſupra ſcriptis, relictis nimirum, ad quas pertinent, poteſtatibus (z^0, $z^{\pm 1}$ $z^{\pm 2}$, ... $z^{\pm n}$) ipſius Radicis (z), quae baſin Syſtematis conſtituit.

Patet autem ſimul ex ipſa comparatione, quomodo diſtantiae Exponentes in Ordinis Exponentes, et contra, converti et inter ſe permutari poſſint.

Varii iſti Coefficientes notandi modi, praeſertim ultimus [c, b], apprime ſunt utiles, ad Coefficientium quaeſitorum ex datis inventionem facilitandam, ſerierumque in his Coefficientibus progreſſum determinandum.

9. *Coefficientes Binomiales* Exponentis ſive Indicis m, notentur *primus*, *ſecundus*, *tertius*, *quartus*, *quintus* ... *ntus*, literis ex ordine $^m\mathfrak{A}$, $^m\mathfrak{B}$, $^m\mathfrak{C}$, $^m\mathfrak{D}$, $^m\mathfrak{E}$ .... $^m\mathfrak{N}$.

$$\text{Ergo } {}^m\mathfrak{A} = \frac{m}{1};\quad {}^m\mathfrak{B} = \frac{m.\ m-1}{1.\ \ 2.};\quad {}^m\mathfrak{C} = \frac{m.\ m-1.\ m-2}{1.\ \ 2.\ \ 3.};\ \ldots$$

$$\ldots\ {}^m\mathfrak{N} = \frac{m.\ m-1.\ m-2.\ m-3.\ m-4.\ \ldots\ m-n+1}{1.\ \ 2.\ \ 3.\ \ 4.\ \ 5\ \ldots\ n};$$

ubi m quilibet poteſt eſſe numerus $\pm\mu$; $\pm\frac{\mu}{\nu}$; $\mu\pm\nu$; etc.

Exponentem m addere literae maiori germanicae, quae Coefficientem exprimit, convenit, praeſertim, ubi in una formula, ut fit interdum, *plures* occurrunt *plurium* Exponentium Coefficientes Binomiales. Quod ſi tamen *idem* manet perpetuo *Exponens* m, hoc omiſſo, ſolae etiam ſcribi poſſunt literae $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ ... v. c.

$$\mathfrak{A}a^{m-1}b + \mathfrak{B}a^{m-2}b^2 + \mathfrak{C}a^{m-3}b^3 + \mathfrak{D}a^{m-4}b^4 \text{ etc.}$$

10. *Coefficientes Polynomiales* [II, 24.] ſive *Tranſpoſitionum Numeri* ſignentur, ſi *una* res, vel *binae*, vel *ternae* vel *quaternae*, vel *quinae* . . dantur permutandae, literis ex ordine minoribus, ſed pariter germanicis $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ .... reſpective.

Hinc $\mathfrak{a}a$; $\mathfrak{b}ab$; $\mathfrak{c}abc$; ſimiliterque, ubi res occurrunt repetitae, $\mathfrak{d}a^2b^2$; $\mathfrak{e}a^3b^2$; $\mathfrak{f}a^3b^2c$; $\mathfrak{g}a^4b^2c$; $\mathfrak{h}a^4b^2cd$; etc, definito ſcilicet Coefficiente Polynomiali, ſive eius ſigno vel litera ex numero rerum, h. e. exponentium particularium ſumma. Valores Coefficientium iſtorum Polynomialium numericos pro ſingulis formis rerum usque decem numero exponit Tabula noſtra VI*ta* (*).

(*) *Infinit. Dignit. p.* 168. 169. *Conf. Tab.* V. *p.* 167. Numeri ipſi Permutationum et Combinationum exhibentur *Tab.* III. *p.* 162–165.

11. *Coefficientes ex Binomialibus*, *Polynomialibus*, ſive etiam *utrisque geniti* ſignentur eo modo, quo ex his ortum trahunt, vel utcunque componuntur addendo, ſubtrahendo, multiplicando, dividendo.

Ingens hic deprehenditur horum Coefficientium varietas, per Compoſitionis modum explicanda. Sic Dignitatis $(1+\alpha z+\beta z^2+\text{etc})^m$ terminus tertius ab Vnitate, h. e. $\dot{C}z^3$, Coefficientem habet fictum $\dot{C} = {}^m\mathfrak{A}a\gamma + {}^m\mathfrak{B}\mathfrak{b}\alpha\beta + {}^m\mathfrak{C}c\alpha^3 = m\gamma + m.\,m-1.\ \alpha\beta + \frac{m.\ m-1.\ m-2}{1.\ \ 2.\ \ 3}\alpha^3$, ubi Coefficiens Binomialis ${}^m\mathfrak{B}$ mutilus $m.\ m-1$ apparet, propter Producti $\alpha\beta$ Coefficientem Polynomialem $\mathfrak{b}=2$, qui multiplicando deſtruit diviſores 1. 2 in ${}^m\mathfrak{B}$ (ψ).

Saepenumero per *ſaltus* progrediuntur Coefficientes, ex numeris figuratis (Tabulae noſtrae III*iae*) definiendi, addendo interdum vel multiplicando etiam augendi, ſubtrahendo vel dividendo minuendi. Apparentis
confuſio. perturbatio
huiusmodi ἀταξίας Exempla, pauca quidem ſed luculenta, exhibuit LAMBERTUS (**); plura uſus ſubminiſtrabit, quibus Tabulae mirificae fines longiſſime prorogantur.

C. *Combinationum cum Permutationibus notationes.*

a) *Operationes combinatoriae.*

12. *Operationes combinatoriae*, ſive Permutationes, Complicationes, Variationes et compoſitae ex his, optime et ſimpliciſſime indicantur, praeponendo ipſa verba, vel literas initiales, literis ſive numeris, poſitis pro rebus, quibuscum Operatio inſtitui debet.

Hinc

Permutationes $(a,\ b,\ c,\ d\ldots.)$ ſive $P(1,\ 2,\ 3,\ 4\ldots.)$;
{ *Complicationes* $(1,\ 2,\ 3,\ 4\ldots)$ vel $C(\alpha,\ \beta,\ \gamma,\ \delta\ldots.)$
{ *Comznationes* $(a, b, c, d\ldots.)$; *Conznationes* $(1, 2, 3, 4\ldots.)$; *etc.* }
Variationes $(1,\ 2,\ 3,\ 4\ldots.)$ ſive $V(\alpha, \beta, \gamma, \delta\ldots)$,

quid ſignificent, obſcurum non eſt.

Sic

(ψ) *Infinit. Dign.* §. XXI. p. 72. Huius generis plures Anomaliae in his Coefficientibus apparent apud *Wolfium Math. El. ſ.* 102. *Anal. Fin.* p. 329.

(**) *Zuſätze zu den logarithm. und trigonom. Tabellen*, §. 112, 113. p. 91-93.

Sic $\frac{Con(n+h)nationes\ (\alpha+\beta+\gamma+\delta\ldots)}{Con(m-k)nationes\ (a+b+c+d\ldots)}$ (a) indicat,

Con (n+h) *nationes* ſuffixarum $\alpha+\beta+\gamma+\delta\ldots$ in *ſummam* collectas, dividendas eſſe cum *ſumma Con* (m-k) *nationum* ex ſuffixis $a+b+c+d\ldots$

Eodem modo per

Species $[(a+b+c\ldots)\ (\alpha+\beta+\gamma\ldots)]$ aut *Sp* $(a+b+c+etc)^n$ *Species* poſtulantur diverſae ſingulae rite ordinatae, vel *Producti* ex $a+b+c+etc$ in $\alpha+\beta+\gamma+$ etc, vel *Dignitatis n* a *Radice* $a+b+c\ldots$, quales per Complexiones determinantur (b) omiſſis nimirum ſimilibus Speciebus [II, 22].

Hinc

Numerus Specierum $(a+b+c\ldots)^4$ ſive $nS(a+b+c\ldots)^4$, *numerum* ſignificat $S(a+b+c\ldots)^4$; ſimiliterque pro aliis huiusmodi notationibus.

a) *Claſſes Complexionum ſimpliciter, cum ſingulis Claſſium Complexionibus.*

13. 'A, 'B, 'C, 'D, ... ſive literae *apicibus indefinitis* notatae maiores, *Unionum*, *Binionum*, *Ternionum*, *Quaternionum*, etc, et reliquarum *Complexionum* rite ordinatarum a literis datis *a*, *b*, *c*, *d*, *e* ..., vel numeris 1, 2, 3, 4, 5 ... ſignificant *Claſſes rite ordinatas*, ex ordine, h. e. Complexionum *Claſſem primam*, *ſecundam*, *tertiam*, *quartam* etc.

Sic

(*a*) Etiam huiusmodi interdum opus eſt notationibus. Huc pertinet Theorema combinatorium Leibnitianum: *Fractio orta ex ſumma Combinationum, diviſa per ſummam Combinationum proxime inferiorum, non poteſt eſſe minor fractione alia ſimiliter facta ex Combinationibus altioribus;* nempe *Fractio ex ſumma binionum, diviſa per ſummam unionum non poterit eſſe minor, quam fractio ex ſumma ternionum, diviſa per ſummam binionum*, et ita porro. Hinc etiam *productum ex ſumma unionum in ſummam ternionum non poteſt eſſe maius quadrato ex ſumma binionum*, et ita ſimiliter in aliis. *Opp. T. III. p. 530.*

(*b*) Exhibuimus Ipſas Species Exponentis ſive Indicis *n* [quae *Coefficientibus Polynomialibus* numeri integri poſitivi *n*, aut *Binomio-polynomialibus* Exponentis *n* valoris cuiusque, auctae, Dignitatis $(a+b+c\ldots)^n$ terminos ſiſtunt] *Tabula I. Infin. Dign. p.* 157, 158. *Conf. ib.* §. X, ubi *1mae*, *2dae*, *3tiae*, *4tae* Dignitatis Species poſitae leguntur 4nomii $a+b+c+d$.

Sic poſitis literis, vel numeris $\left\{\begin{matrix}1, & 2, & 3\\ a, & b, & c\end{matrix}\right\} = {}'A$, fit $\left\{\begin{matrix}12, & 13, & 23\\ ab, & ac, & bc\end{matrix}\right\} = {}'B$,

et $\left\{\begin{matrix}123\\ abc\end{matrix}\right\} = {}'C$, Claſſibus nimirum hic cum tertia deſinentibus, ubi res non admittuntur repetitae; contra vero pergentibus, cum ipſarum Complexionibus, in infinitum [II, 19]. Dicuntur autem 'A, 'B, 'C etc *Claſſes Complexionum ſimpliciter*, ut diſtinguantur ab his, quarum Complexiones ſingulae producunt *numerum definitum* ſive datum n.

14. $'A1$, $'A2 \ldots {}'B3 \ldots {}'C4 \ldots {}'Dn, \ldots$ appoſito ſcilicet ad dextram literae, quae Claſſem indicat, numero 1, 2, 3, 4 ... n, *Claſſis* ſignificant expoſitae et *rite ordinatae Complexiones* ſingulas, rite ordinatas, *primam*, *ſecundam*, *tertiam*, *quartam* .... *ntam*.

Ergo [13] $'A3 = 3 = c$; $'B2 = 13 = ac$; $'C1 = 123 = abc$; ſimiliterque ubi res admittuntur repetitae.

15. $\mathfrak{a}'A$, $\mathfrak{b}'B \ldots \mathfrak{c}'C1$, $\mathfrak{d}D'2 \ldots \mathfrak{e}'En \ldots$, literae nempe germanicae minores, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \ldots$ *Claſſibus homonymis* integris, 'A, 'B, 'C, 'D ... ſive *Claſſium definitis Complexionibus* 'A1, 'A2 ... 'B3 ... 'C4 .. praefixae, notant, Claſſium vel Complexiones omnes, vel definitas tantum ipſarum Species, Coëfficientibus polynomialibus augendas eſſe [10]. Interdum etiam praefiguntur ſimul Coefficientes Binomiales $^m\mathfrak{A}$, $^m\mathfrak{B}$, $^m\mathfrak{C}$ ...

Hinc [13] $\mathfrak{c}'C = 6abc$; $\mathfrak{b}'B3 = 2bc$; ſimiliterque

$$^m\mathfrak{B}\,\mathfrak{b}\,'B = \frac{m.\ m-1}{1.\ \ 2}(2ab + 2ac + 2bc);\ \text{et}\ ^m\mathfrak{B}\,\mathfrak{b}\,'B1 = m.\,m-1.\ ab\ [11]$$

c) *Claſſes Complexionum numeri propoſiti n, cum ſingulis Claſſium Speciebus.*

16. ^nA, ^nB, $^nC \ldots {}^nN$, *Unionum*, *Binionum*, *Ternionum* .... *Ntionum* ex numeris 1, 2, 3, 4, 5 ... provenientium et rite ordinatarum, ſignificant *Claſſes rite ordinatas*, quarum Complexiones numericae ſingulae *numerum producunt* propoſitum n.

Sic poſito $n = 5$, fit [II, 32]

$$^5A = 5;\ ^5B = \left\{\begin{matrix}14\\ 23\end{matrix}\right\};\ ^5C = \left\{\begin{matrix}113\\ 122\end{matrix}\right\};\ ^5D = 1112;\ ^5E = 11111$$

Ergo

$$\text{Ergo pro ſeriebus}\left\{\begin{array}{l} 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad 8, + \text{etc} \dots \text{etc} \\ \alpha + \beta + \gamma + \delta + \varepsilon + \zeta + \eta + \vartheta + \text{etc} = p \\ \alpha z^{m} + \beta z^{m+d} + \gamma z^{m+2d} + \delta z^{m+3d} + \varepsilon z^{m+4d} + \text{etc} = x \\ \alpha z^{a} + \beta z^{b} + \gamma z^{c} + \delta z^{d} + \varepsilon z^{e} + \text{etc} = y \end{array}\right\}$$

in quibus numeris 1, 2, 3, 4, 5 .... ex ordine,
literae ſeu Coefficientes α, β, γ, δ, ε .... reſpondent, fit:

$$^{5}A = \varepsilon,\ ^{5}B = \begin{bmatrix} \alpha\delta \\ \beta\gamma \end{bmatrix};\ ^{5}C = \begin{bmatrix} \alpha\alpha\gamma \\ \alpha\beta\beta \end{bmatrix};\ ^{5}D = \alpha\alpha\alpha\beta;\ ^{5}E = \alpha\alpha\alpha\alpha\alpha.$$

17. $^{n}B1$, $^{n}B2$, $^{n}B3$ ... ^{n}Bm, ſimiliterque ^{n}Cm, ^{n}Dm ... ^{n}Nm, ſingulae notantur ſingularum *Claſſium rite ordinatarum Complexiones* rite ordinatae, numerum producentes n, ex ordine *prima*, *ſecunda*, *tertia* .... *mta*.

Sic poſito rurſus $n = 5$, fit [16] $^{5}B1 = 14 = \alpha\delta$;
$^{5}B2 = 23 = \beta\gamma$; $^{5}C1 = 113 = \alpha\alpha\gamma$; $^{5}C2 = 122 = \alpha\beta\beta$.

Sic $^{4}B1 = 13 = \alpha\gamma$; et $^{7}D1 = 1114 = \alpha^{3}\delta$; $^{7}D2 = 1123 = \alpha^{2}\beta\gamma$;
$^{7}D3 = 1222 = \alpha\beta^{3}$; ſimiliterque pro aliis aliarum Claſſium Complexionibus numerorum, et quae inde determinantur, literarum.

18. $^{n}Az^{*}$, $^{n}Bz^{*}$, $^{n}Cz^{*}$, $^{n}Dz^{*}$ ... poſtulant, ut Seriei datae *Exponentes* literae *z particulares*, pro ſingulis Claſſium Complexionibus *addantur* eodem modo omnes, quo *Coefficientes*, qui reſpondent, in his Complexionibus *multiplicantur*. Idem intellige, ubi Claſſium definitis Complexionibus ^{n}Bm, ^{n}Cm ... iungitur z^{*}; v. c. $^{n}Bmz^{*}$, $^{n}Cmz^{*}$ ....

Sic, poſita Serie maxime generali,

$$\alpha z^{a} + \beta z^{b} + \gamma z^{c} + \delta z^{d} + \varepsilon z^{e} + \text{etc} = y$$

Seriei y reſpondent,

Coefficientibus	α, β, γ, δ, ε	ex ordine,
Exponentes	a, b, c, d, e	homonymi
et *numeri*	1, 2, 3, 4, 5	ambobus.

Ergo [16]

$$^{5}Bz^{*} = [14 + 23]z^{*} = \alpha\delta z^{a+d} + \beta\gamma z^{b+c}$$
$$^{5}Cz^{*} = [113 + 122]z^{*} = \alpha^{2}\gamma z^{2a+c} + \alpha\beta^{2} z^{a+2b}$$

Similiterque [17]

$$^{7}D1z^{*} = [1114]z^{*} = \alpha^{3}\delta z^{3a+d}$$
$$\text{et}\ ^{7}D2z^{*} = [1123]z^{*} = \alpha^{2}\beta\gamma z^{2a+b+c}$$

verbo, ſi

$$^{\mu}\mathfrak{Tn}\ ^{n}Nmz^{*} = {}^{\mu}\mathfrak{Tn}\,[\alpha^{m-n}\beta^{p}\gamma^{q}\delta^{r}\varepsilon^{s}\dots]z^{*} = T,$$
$$\text{fit}\ T = {}^{\mu}\mathfrak{Tn}\,\alpha^{m-n}\beta^{p}\gamma^{q}\delta^{r}\varepsilon^{s}\dots z^{(m-n)a+pb+qc+rd+se\dots}$$

Coefficientes nempe Binomiales ${}^{\mu}\mathfrak{N}$ atque etiam Polynomiales $\mathfrak{n}$, qui interdum in valoribus termini T occurrunt ſimul, Exponentes literae z nullo modo afficiunt (c).

19. $\mathfrak{a}^n A$, $\mathfrak{b}^n B$, $\mathfrak{c}^n C$, $\mathfrak{d}^n D \ldots {}^{\mu}\mathfrak{N}^n N$, itemque ${}^m\mathfrak{A}\,\mathfrak{a}^n A$, ${}^m\mathfrak{B}\,\mathfrak{b}^n B$, ${}^m\mathfrak{C}\,\mathfrak{c}^n C \ldots$ et ${}^m\mathfrak{B}\,\mathfrak{b}^n B1$, ${}^{\mu}\mathfrak{C}\,\mathfrak{c}^n C2$, ${}^{\frac{m}{\nu}}\mathfrak{D}\,\mathfrak{d}^n D3 \ldots$ *Claſſium* ſignificant expoſitarum *Complexiones*, vel *omnes*, vel ſingulas *definitas*, Coefficientibus numericis *Polynomialibus*, *Binomialibus* multiplicandas eſſe.

Sic $\mathfrak{a}^5 A = 1\varepsilon$; $\mathfrak{b}^5 B = 2\alpha\delta + 2\beta\gamma$; $\mathfrak{c}^5 C = 3\alpha^2\gamma + 3\alpha\beta^2$; etc.

${}^{\mu}\mathfrak{C}\,\mathfrak{c}^5 C = \frac{\mu . \mu-1 . \mu-2}{1 . 2 . 3}[3\alpha^2\gamma + 3\alpha\beta^2]$; ſimiliterque

$\mathfrak{c}^7 C2 z^{*} = 6\alpha\beta\delta z^{a+b+c}$; et ${}^{\frac{m}{\nu}}\mathfrak{D}\,\mathfrak{d}^7 D2 = \frac{\frac{m}{\nu} . \frac{m-1}{\nu} . \frac{m-2}{\nu} . \frac{m-3}{\nu}}{1 . 2 . 3 . 4} . 12\alpha^2\beta\gamma.$

nempe Coefficientes polynomiales, ſive transpoſitionum numeri, per Coefficientium $\alpha, \beta, \gamma, \delta, \varepsilon \ldots$ *Exponentes particulares* determinantur (d).

20. $\mathfrak{a}^n A . \mathfrak{c}^m C$ et $\mathfrak{b}^n B . \mathfrak{d}^r D . \mathfrak{k}^s K$; etc, itemque $\mathfrak{b}^n Bm . \mathfrak{g}^h Gr$; et $\mathfrak{e}^r Ef . \mathfrak{h}^n Hr . \mathfrak{l}^s Lk$; etc, iubent *Claſſes* expoſitas *integras*, vel *Claſſium definitas Complexiones*, ſed *unius* eiusdemque *Seriei* omnes, in ſe ducendas [II, 34] et Coefficientibus, qui reſpondent, Polynomialibus multiplicandas eſſe.

Nempe apparent hic *Complexionum Complexiones*, ſive *Complexiones ſecundi*, *tertii* etc Ordinis, ſed *eiusdem* omnes Seriei unius p; quapropter opus non eſt Claſſi addere notam p, ut fieri debet, ubi plurium Serierum $p, q, r \ldots$ occurrunt plures Claſſes, uti mox videbimus.

21. $\overset{p}{\mathfrak{a}^n A} . \overset{q}{\mathfrak{c}^m C}$ et ${}^{\mu}\mathfrak{B}\,\overset{p}{\mathfrak{b}^n B} . \overset{q}{\mathfrak{d}^r D} . \overset{r}{\mathfrak{k}^s K}$; itemque $\overset{p}{\mathfrak{b}^n Bm} . \overset{q}{\mathfrak{g}^h Gr}$; et ${}^{\mu}\mathfrak{E}\,\overset{p}{\mathfrak{e}^r Ef} . \overset{q}{\mathfrak{h}^n Hr} . \overset{r}{\mathfrak{l}^s Lk}$; etc ſignificant, *Claſſes* expoſitas *integras*, ſive *Claſſium definitas Complexiones*, ſed *diverſarum Serierum*, $p, q, r \ldots$, (quarum ſigna in Claſſium vertice conſpicua) omnes, una cum Coefficientibus polynomialibus, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \ldots$ aut etiam ſimul Binomialibus ${}^{\mu}\mathfrak{B}$, ${}^{\mu}\mathfrak{C} \ldots$ in ſeſe ducendas eſſe.

 Termi-

(c) *Conf. Infinit. Dignit.* §. XXVIII, *ſ.* 10, 11. *p.* 149, 150.
(d) *Infinit. Dignit.* §. XIII. *ſ.* 2, 3 et 6. *p.* 31-35.

Termini huiusmodi Complexionum *ſuperiorum* ordinum, a Seriebus diverſis $p, q, r \ldots$ oriundarum, in Serierum Inverſione ſive Regreſſu et alibi frequentes occurrunt. Sic v. c. functionis fractae,

$$\frac{(1 + az + bz^2 + cz^3 + dz^4 + ez^5 + etc)^m}{(1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc})^\mu} = \frac{p^m}{q^\mu} = p^m q^{-\mu}$$

absque terminorum recurrentium incommodis, in terminis ſtatim ſimpliciſſimis evoluendae, termini extra ordinem *tertii* Coefficientem reperimus (*)

$$\begin{array}{l} \overset{p}{1}.\,[{}^{-\mu}\mathfrak{A}\alpha^3\overset{q}{A} + {}^{-\mu}\mathfrak{B}\beta^3\overset{q}{B} + {}^{-\mu}\mathfrak{C}\gamma^3\overset{q}{C}] \\ + {}^{m}\mathfrak{A}a^1\overset{p}{A} \cdot [{}^{-\mu}\mathfrak{A}\alpha^2\overset{q}{A} + {}^{-\mu}\mathfrak{B}\beta^2\overset{q}{B}] \\ + [{}^{m}\mathfrak{A}a^2\overset{p}{A} + {}^{m}\mathfrak{B}b^2\overset{p}{B}] \cdot {}^{-\mu}\mathfrak{A}\alpha^1\overset{q}{A} \\ + [{}^{m}\mathfrak{A}a^3\overset{p}{A} + {}^{m}\mathfrak{B}b^3\overset{p}{B} + {}^{m}\mathfrak{C}c^3\overset{p}{C}].\,\overset{q}{1} \end{array}$$

nempe Producta hic poſtulantur ex Claſſium plurium ſummis, a Seriebus pendentibus diverſis p et q. Namque diverſitas hic obſervatur tantum non infinita.

d) *Claſſes Variationum ſimpliciter, ſive Complicationes ſimul cum Permutationibus.*

22. Claſſes *Variationum* ſimpliciter ſimiliter ut *Complexionum* literis maioribus, ſed *Italicis* deſignentur, apicibus indefinite notatis, ut

$'A$, $'B$, $'C$, etc quibus Complexionum ſimul cum Permutationibus Claſſes, *prima*, *ſecunda*, *tertia*, etc, *Unionum* ſcilicet, *Binionum*, *Ternionum* etc indicantur.

e) *Claſſes Variationum numeri propoſiti* n, *cum ſingulis Claſſium ſpeciebus.*

23. ^nA, ^nB, ^nC, $^nD \ldots {}^nN$; Numeri dati n, et numerorum, a quibuslibet *binis*, *ternis*, *quaternis* ... numeris minoribus 1, 2, 3, 4 ... $n-1$, varie ſimul permutatis oriundorum, *Claſſes* ſignificant *primam*, *ſecundam*, *tertiam*, *quartam* ... *n*tam, quarum ſpecies numericae ſingulae numerum addendo *producunt propoſitum* n.

Sic

(*) *Infinit. Dignit.* §. XXVI. 1, 3, 4, 6. Signa $\alpha^3 A$, ... $\beta^3 B$ ... $c^3 C$ ... explicat Tabula noſtra Vta; ibid. p. 167.

Sic v. c. poſito $n = 4$, fit

$$ {}^4A = \overset{p}{4};\quad {}^4B = \begin{Bmatrix} \overset{q}{1}, & \overset{p}{3} \\ 2, & 2 \\ \underset{q}{3}, & \underset{p}{1} \end{Bmatrix};\quad {}^4C = \begin{Bmatrix} \overset{r}{1}, & \overset{q}{1}, & \overset{p}{2} \\ 1, & 2, & 1 \\ \underset{r}{2}, & \underset{q}{1}, & \underset{p}{1} \end{Bmatrix};\quad {}^4D = \overset{s}{1}, \overset{r}{1}, \overset{q}{1}, \overset{p}{1}; $$

Igitur ${}^nA$, ${}^nB$, ${}^nC$ etc, ab ${}^n\mathrm{A}$, ${}^n\mathrm{B}$, ${}^n\mathrm{C}$ etc [16] non differunt, niſi quod horum numeri Complexionum pro illis ſimul permutentur; ut adeo vere ſint *Variationes numeri propoſiti.* Sed quia numeri 1, 2, 3, 4 . . . plurium Serierum p, q, r, s . . . Coefficientes diverſos hic ex ordine deſignant (namque hoc niſi eſſet, *Species* eorundem ſed permutatorum numerorum ut 1, 3 et 3, 1; *ſimiles* forent et *indifferentes*) neceſſe fuit, Series numerorum verticales literis ſupra vel infra ſcriptis notare, ut conſtet, ad cuius Seriei Coefficientes numeri ſinguli pertineant.

Sic v. c.

$$ \text{poſitis } \textit{numeris}\ \ 1,\qquad 2,\qquad 3,\qquad 4,\qquad 5\ \ldots $$

$$ \text{et } \textit{Seriebus} \begin{Bmatrix} az + bz^2 + cz^3 + dz^4 + ez^5 \ldots = p \\ \mathrm{A}z + \mathrm{B}z^2 + \mathrm{C}z^3 + \mathrm{D}z^4 + \mathrm{E}z^5 \ldots = q \\ \mathfrak{a}z + \mathfrak{b}z^2 + \mathfrak{c}z^3 + \mathfrak{d}z^4 + \mathfrak{e}z^5 \ldots = r \\ \mathfrak{A}z + \mathfrak{B}z^2 + \mathfrak{C}z^3 + \mathfrak{D}z^4 + \mathfrak{E}z^5 \ldots = s \\ \text{etc} \qquad\qquad \text{etc} \qquad\qquad \text{etc} \end{Bmatrix}. $$

fit, poſitis pro numeris literis ſive Coefficientibus,

$$ {}^4A = \mathrm{d};\quad {}^4B = \begin{Bmatrix} \mathrm{A\,c} \\ \mathrm{B\,b} \\ \mathrm{C\,a} \end{Bmatrix};\quad {}^4C = \begin{Bmatrix} \mathfrak{a}\,\mathrm{A\,b} \\ \mathfrak{a}\,\mathrm{B\,a} \\ \mathfrak{b}\,\mathrm{A\,a} \end{Bmatrix};\quad {}^4D = \mathfrak{A}\,\mathfrak{a}\,\mathrm{A\,a}; $$

namque ſiſtunt Coefficientium producta ex ordine Serierum

$$ \begin{array}{lll} pq, \textit{ Claſſes Binionum} & {}^2B,\ {}^3B \ldots & {}^{n+1}B \\ pqr, \textit{ — Ternionum} & {}^3C,\ {}^4C \ldots & {}^{n+2}C \\ pqrs, \textit{ — Quaternionum} & {}^4D,\ {}^5D \ldots & {}^{n+3}D \\ \textit{etc} & \textit{etc} & \textit{etc} \end{array} $$

numerorum definitorum, 2, 3, 4, 5 . . . n, $n+1$, $n+2$, $n+3$, et qui per ipſos deſignantur, Coefficientium. Ex Claſſibus autem his colliguntur, quarum Complexiones *numerum,* addendo partes ſingulas, efficiunt *propoſitum n*; h. e. ${}^nA$, ${}^nB$, ${}^nC$, etc.

24. ${}^nB1$, ${}^nB2$, ${}^nB3$ . . . ${}^nBm$, ſimiliterque ${}^nCm$, ${}^nDm$ . . . ${}^nNm$, *Claſſium* expoſitarum *definitas Species* ſignificant *primam*, *ſecundam*, *tertiam*,

tiam, ... mtam. Nempe etiam hic intelliguntur Claſſes Variationum ſite ordinatae.

Sic poſito $n = 4$, fit [23]

$$ {}^4B1 = \overset{q\ p}{1\ 3} = \text{Ac};\quad {}^4C2 = \overset{r\ q\ p}{1\ 2\ 1} = \text{aBa};$$
$$\text{itemque } {}^4D7 = 2112 = \mathfrak{B}\text{aAb};$$

25. ${}^nAz^*$, ${}^nBz^*$, ${}^nCz^*$ . . . itemque ${}^nBm.\ z^*$, ${}^nCm.\ z^*$ . . . . indicant, Serierum datarum, *p, q, r, s* . . . *Exponentes* literae *z particulares*, pro Claſſium Speciebus vel omnibus, vel definitis certo numero *m*, *addendos* eſſe eodem modo omnes, quo Serierum *Coefficientes* homonymi *multiplicantur*.

Sic

poſitis *numeris* 1, 2, 3, 4, 5, ...

$$\text{et } \textit{Seriebus} \begin{bmatrix} \mathfrak{A}z^a + \mathfrak{B}z^b + \mathfrak{C}z^c + \mathfrak{D}z^d + \mathfrak{E}z^e + \text{etc} = p \\ \text{A}z^a + \text{B}z^b + \text{C}z^c + \text{D}z^d + \text{E}z^e + \text{etc} = q \end{bmatrix}$$

erit

$$ {}^4Bz^* = (\overset{q\ p}{1\ 3} + \overset{q\ p}{2\ 2} + \overset{q\ p}{3\ 1})\ z^* = \text{A}\mathfrak{C}z^{a+c} + \text{B}\mathfrak{B}z^{b+b} + \text{C}\mathfrak{A}z^{c+a}$$
$$\text{hinc } {}^4B1.z^* = \text{A}\mathfrak{C}z^{a+c};\quad \text{et } {}^4B3.\ z^* = \text{C}\mathfrak{A}z^{c+a}$$

Similiterque in reliquis caſibus. Etiam hic iuncti reperiuntur interdum Coefficientes *Polynomiales* a, b, c . . et *Binomiales* ${}^m\mathfrak{A}$, ${}^m\mathfrak{B}$, ${}^m\mathfrak{C}$ . . . qui pariter Exponentes nullo modo afficiunt [18].

26. Quia *figurae numericae ſingulae* Specierum indefinitarum in Claſſibus ${}^nA$, ${}^nB$, ${}^nC$, ${}^nD$ . . . ., et proinde etiam definitarum ${}^nBm$, ${}^nC\mu$, ${}^nD\nu$ . . . Serierum plurium diverſarum *p, q, r, s* . . . Coefficientes ſive etiam terminos deſignant [23], *unius p* in ${}^nA$, *duarum qp* in ${}^nB$, *trium rqp* in ${}^nC$, *quatuor srqp* in ${}^nD$ etc, poſſunt etiam ipſis Claſſibus iungi in vertice Serierum notae *p, q, r, s* . . . hoc modo $\overset{p}{{}^nA}$, $\overset{qp}{{}^nB}$, $\overset{rqp}{{}^nC}$, $\overset{srqp}{{}^nD\mu}$ . . . $\overset{\ldots rqp}{{}^nN\nu}$, ubi *N* Claſſem ſignificat indefinitam, et . . . *rqp* Series numero indefinitas tot, quot unitates habet *numerus* Claſſis *N*. Interdum fit, ut Claſſibus ${}^nA$, ${}^nB$, ${}^nC$ . . . non reſpondeant Series *p, qp, rqp* . . . reſpective, ſed alio ordine, v. c. $\overset{p}{{}^nA}$, $\overset{rq}{{}^nB}$, $\overset{tsr}{{}^nC}$, $\overset{ruts}{{}^nD}$ . . .; unde opportunitas huiusmodi notationis apparet.

Rerum datarum ſigna, *a, b, c, d* . . . (quibus ſaepiſſime Coefficientes terminorum in Seriebus, interdum etiam termini ipſi indicantur) cum reliquis, quae expoſita ſunt inde a 13-25, *Complexionum ſingularum* atque *Claſſium*

Claſſium ſignis, ad *Elementa* pertinent *combinatoria* [II, 38]; quae, vel ſola illa, vel intermixtis etiam aliis quantitatibus aut numeris, per praecepta analytica problematum conditionibus adaptantur.

Elementa haec combinatoria, quem habeant uſum, indicavimus [II, 38]. Commode autem poſſunt, utpote elementa ſimpliciſſima et maxime univerſalia, per tabulas disponi, eo modo, quo in Appendice ad Methodum Potentiarum oſtendi, tabulis deſcribendis decem numero, quarum, non ſolum in problematis proprie ſic dictis combinatoriis, ſed etiam in Analyſi univerſa et Arithmetica numerorum, (quorum ipſae figurae ſingulae, tanquam Coefficientes poteſtatum denarii in computum hic adhibentur) aeque late patet uſus, ac tabularum ſinuum, logarithmorum, aliorumque numerorum in trigonometria aut geometria practica.

§. VI.

Variae Signorum relationes.

1. *Complexiones ſimpliciter* ad *Complexiones numeri definiti* eam habent relationem, ut Totum ad Partes; nam in illis ordo tantum reſpicitur, ſecundum quem res propoſitae, literae numeri, coniunguntur; in his, praeter ordinem, ſimul etiam ſumma numerorum pro rebus datis ſingulis poſitorum. Hinc Claſſes Complexionum ſimpliciter comparando cum Claſſibus Complexionum numerorum definitorum, apparet, fore ſeorſim:

$$
\begin{array}{llllllll}
{}'A = & {}^1A + {}^2A + {}^3A + {}^4A + {}^5A \;\ldots\ldots + {}^nA + {}^{n+1}A \\
{}'B = & {}^2B + {}^3B + {}^4B + {}^5B \;\ldots\ldots + {}^nB + {}^{n+1}B \\
{}'C = & {}^3C + {}^4C + {}^5C \;\ldots\ldots + {}^nC + {}^{n+1}C \\
{}'D = & {}^4D + {}^5D \;\ldots\ldots + {}^nD + {}^{n+1}D \\
\text{etc} & \qquad\qquad \text{etc} \quad \text{etc} \qquad \text{etc} \\
{}'N = & {}^nN + {}^{n+1}N
\end{array}
$$

et coniunctim:

$$
\begin{aligned}
{}'A + {}'B + {}'C + {}'D + \text{etc} = {}^1A &+ ({}^2A + {}^2B) + ({}^3A + {}^3B + {}^3C) \\
&+ ({}^4A + {}^4B + {}^4C + {}^4D) + \text{etc}
\end{aligned}
$$

2. Eodem modo, *Variationes ſimpliciter* comparando cum *Variationibus numerorum definitorum*, reperitur ſeorſim:

$$\overset{p}{'A} = \overset{p}{{}^1A} + \overset{p}{{}^2A} + \overset{p}{{}^3A} + \overset{p}{{}^4A} \ldots\ldots + \overset{p}{{}^nA}$$

$$\overset{qp}{'B} = \overset{qp}{{}^2B} + \overset{qp}{{}^3B} + \overset{qp}{{}^4B} \ldots\ldots + \overset{qp}{{}^nB}$$

$$\overset{rqp}{'C} = \overset{rqp}{{}^3C} + \overset{rqp}{{}^4C} \ldots\ldots + \overset{rqp}{{}^nC}$$

etc etc etc

$$\overset{\ldots srqp}{'N} = \overset{\ldots srqp}{{}^nN}$$

et coniunctim:

$$\overset{p}{'A} + \overset{qp}{'B} + \overset{rqp}{'C} + \overset{srqp}{'D} + \text{etc} = \overset{p}{{}^1A} + (\overset{p}{{}^2A} + \overset{qp}{{}^2B}) + (\overset{p}{{}^3A} + \overset{qp}{{}^3B} + \overset{rqp}{{}^3C})$$
$$+ (\overset{p}{{}^4A} + \overset{qp}{{}^4B} + \overset{rqp}{{}^4C} + \overset{srqp}{{}^4D}) + \text{etc}$$

ſimiliterque ubi Variationum Claſſibus ex ordine ${}^nA$, ${}^nB$, ${}^nC$, ${}^nD$ .... ${}^nN$ non reſpondent Serierum producta p, qp, rqp, srqp, ... et ... tsrqp, reſpective, ſed alio ordine [V, 26.]

3. In problematibus circa Series, vel una tantum proponitur Series $\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc} = p$, vel plures diverſae, p, q, r, s, etc. *Illius* igitur ſi ponas in promtu eſſe *Dignitates* ex ordine p^1, p^2, p^3, p^4, p^5, etc (quibus ſimul reliquarum q, r, s ... Dignitates repraeſentantur), *harum* autem *Producta* pq, pqr, $pqrs$, $pqrst$ etc quaelibet: illico apparet, Problemata circa Series propoſita (f) vel ab *illius* Dignitatibus, vel *harum* Productis, vel utrisque pendere, ſolutionemque abſolvi poſſe, ſi conſtet, *qui termini* Dignitatum aut Productorum Serierum p, q, r, s, etc, et *quomodo*, formulas quaeſitas ingrediantur.

4. Terminos Dignitatum $p^1, p^2, p^3 \ldots p^n$ et Productorum $pq, pqr, pqrs, pqrs \ldots$ exhibui (g), Elementorum combinatoriorum ope, h. e. ad uſum maxime accommodate, ubi nempe termini deſiderantur, per variabilis z poteſtates et coefficientes datos ſimpliciſſime expreſſi. Supereſt ut *Elementorum* iſtorum *relationes* ad terminos ipſos ſive terminorum Coefficientes *qui* ſcilicet ſint *numero*, et *cuius Dignitatis* aut *Producti*, breviter nunc indicetur.

2) *Ter-*

(f) *Conf.* §. III. p. XXVI.

(g) Illos nimirum *Infinit. Dignit.* §. XXIII. *ſ.* 8; hos autem *Ibid.* §. XXVII. *ſ.* 12.

a) Terminorum aut Coefficientium in Dignitatibus $p^1, p^2, p^3 \ldots p^n$ relationes ad Elementa combinatoria; et viciſſim.

5. Oſtendimus (h) reperiri $p^1\rceil n = \mathfrak{a}^n A z^n$; $p^2\rceil(n-1) = \mathfrak{b}^n B z^n$; $p^3\rceil(n-2) = \mathfrak{c}^n C z^n$; $p^4\rceil(n-3) = \mathfrak{d}^n D z^n$; $p^5\rceil(n-4) = \mathfrak{e}^n E z^n$; ..... et $p^\nu\rceil(n-\nu+1) = \mathfrak{n}^n N z^n$, et Claſſem indefinitam N, cum Coefficiente polynomiali $\mathfrak{n}$, a determinatione Exponentis indefiniti ν unice pendere, ſic ut, poſito $\nu = 1, 2, 3, 4, 5, 6 \ldots$. ſucceſſive, fiat $N = A, B, C, D, E, F \ldots$; et $\mathfrak{n} = \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f} \ldots$. reſpective. Hinc

$\mathfrak{a}^1 A z^1 = p^1\rceil 1$				
$\mathfrak{a}^2 A z^2 = p^1\rceil 2$	$\mathfrak{b}^2 B z^2 = p^2\rceil 1$			
$\mathfrak{a}^3 A z^3 = p^1\rceil 3$	$\mathfrak{b}^3 B z^3 = p^2\rceil 2$	$\mathfrak{c}^3 C z^3 = p^3\rceil 1$		
$\mathfrak{a}^4 A z^4 = p^1\rceil 4$	$\mathfrak{b}^4 B z^4 = p^2\rceil 3$	$\mathfrak{c}^4 C z^4 = p^3\rceil 2$	$\mathfrak{d}^4 D z^4 = p^4\rceil 1$	
etc : etc	etc etc	etc etc	etc etc	etc
$\mathfrak{a}^n A z^n = p^1\rceil n$	$\mathfrak{b}^n B z^n = p^2\rceil(n-1)$	$\mathfrak{c}^n C z^n = p^3\rceil(n-2)$	$\mathfrak{d}^n D z^n = p^4\rceil(n-3)$	etc

Similiterque $\mathfrak{e}^n E z^n = p^5\rceil(n-4)$; $\mathfrak{f}^n F z^n = p^6\rceil(n-5)$; $\mathfrak{g}^n G z^n = p^7\rceil(n-6)$; etc, etc.

unde, quia [V, 2] *Coefficiens termini* $p^\nu\rceil n = p^\nu \varkappa n$, fit

$\mathfrak{a}^1 A = p^1 \varkappa 1$				
$\mathfrak{a}^2 A = p^1 \varkappa 2$	$\mathfrak{b}^2 B = p^2 \varkappa 1$			
$\mathfrak{a}^3 A = p^1 \varkappa 3$	$\mathfrak{b}^3 B = p^2 \varkappa 2$	$\mathfrak{c}^3 C = p^3 \varkappa 1$		
$\mathfrak{a}^4 A = p^1 \varkappa 4$	$\mathfrak{b}^4 B = p^2 \varkappa 3$	$\mathfrak{c}^4 C = p^3 \varkappa 2$	$\mathfrak{d}^4 D = p^4 \varkappa 1$	
etc etc	etc etc	etc etc	etc etc	etc etc
$\mathfrak{a}^n A = p^1 \varkappa n$	$\mathfrak{b}^n B = p^2 \varkappa (n-1)$	$\mathfrak{c}^n C = p^3 \varkappa (n-2)$	$\mathfrak{d}^n D = p^4 \varkappa (n-3)$	etc etc

et proinde $\mathfrak{e}^n E = p^5 \varkappa (n-4)$; $\mathfrak{f}^n F = p^6 \varkappa (n-5)$; $\mathfrak{g}^n G = p^7 \varkappa (n-6)$; et ſic porro.

Data igitur formula, quae indicat, ex quibus terminis $\rceil$ vel t, ſive coefficientibus $\varkappa$ vel c, potentiarum $p^1, p^2, p^3, p^4 \ldots$. Seriei elementaris $\alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \varepsilon z^5 + \text{etc} = p$, Series data ſive quaeſita conſtet, ſtatim inde poterit formula, per Elementa combinatoria $\mathfrak{a}^n A$, $\mathfrak{b}^n B$, $\mathfrak{c}^n C$, etc expreſſa, et contra, data hac, illa deduci, per Dignitatum terminos.

EXEMPLUM I. Seriei $1 + \alpha z + \beta z^2 + \gamma z^3 + \delta z^4 + \text{etc} = 1 + p$, determinatur (i) Dignitatis $(1+p)^m$ terminus $n$tus inde a primo 1, hac formula:

$$\Gamma^{(n)} z^n = {}^m\mathfrak{A} p^1\rceil n + {}^m\mathfrak{B} p^2\rceil(n-1) + {}^m\mathfrak{C} p^3\rceil(n-2) + {}^m\mathfrak{D} p^4\rceil(n-3) + \text{etc}$$

unde, poſito v. c. exponente $n = 4$, ſcriptis autem, pro Dignitatum terminis expoſitis, Elementis ſecundum relationem mox traditam, producitur

 $\Gamma^{(4)}$

(h) *Infinit. Dignit.* §. XXIII, ſ. 8.
(i) *Infinit. Dignit.* §. XXI. ſ. 3.

$$\Gamma^{(4)}z^4 = ({}^m\mathfrak{A}\mathfrak{a}^4A + {}^m\mathfrak{B}\mathfrak{b}^4B + {}^m\mathfrak{C}\mathfrak{c}^4C + {}^m\mathfrak{D}\mathfrak{d}^4D)z^4 \quad (^k)$$

et hinc, tranſcriptis valoribus Elementorum $\mathfrak{a}^4A$, $\mathfrak{b}^4B$, $\mathfrak{c}^4C$, $\mathfrak{d}^4D$ ex Tabula (^{l}).

$$\Gamma^{(4)}z^4 = \left(m\delta + \frac{m.\,m-1}{1.\quad 2}[2\alpha\gamma + \beta^2] + \frac{m.\,m-1.\;m-2}{1.\quad 2.\quad 3}\cdot 3\alpha^2\beta + \frac{m.\,m-1.\;m-2.\;m-3}{1.\quad 2.\quad 3.\quad 4}\alpha^4\right)z^4.$$

EXEMPLUM II. Poſito *Logar. hyp.* $\frac{1+\alpha z+\beta z^2\ldots}{1-\alpha z-\beta z^2\ldots} = l\,h\frac{1+p}{1-p} = S$, reperitur (^{m})

$$S = 2\mathfrak{a}^1Az + 2\mathfrak{a}^2Az^2 + (2\mathfrak{a}^3A + \tfrac{2}{3}\mathfrak{c}^3C)z^3 + (2\mathfrak{a}^4A + \tfrac{2}{3}\mathfrak{c}^4C)z^4 + \text{etc}$$

$$\text{Ergo } S = 2p^1\varkappa 1.\,z + 2p^1\varkappa 2.\,z^2 + (2p^1\varkappa 3 + \tfrac{2}{3}p^3\varkappa 1)z^3 + (2p^1\varkappa 4 + \tfrac{2}{3}p^3\varkappa 2)z^4 + \text{etc}$$

unde patet, *quibus* terminis terminorumque Coefficientibus *Dignitatum* p^1, p^2, p^3 ..., conſtet Series S, quae ſtatim poterit, ubi opus eſt, per ipſos Coefficientes datos α, β, γ, δ, ε .... exprimi, tranſcriptis valoribus Elementorum $\mathfrak{a}^nA$, $\mathfrak{c}^nC$, $\mathfrak{e}^nE$, $\mathfrak{g}^nG$, etc ex Tabula (l).

b) *Terminorum aut Coefficientium a Serierum Productis p, pq, pqr, pqrs, pqrst, ... et pqrst ... relationes ad Elementa combinatoria; et contra.*

6. Similiter, poſitis Seriebus elementaribus pluribus diverſis p, q, r, s .... numero ν, ex producti $pqrs$ ... termino generali (^{n}) ſive $pqrs\ldots\,{}^{\mathsf{J}}\mu = {}^{\mu+\nu-1}Nz^{\mu+\nu-1}$, (ubi Claſſis indefinita N per factorum $pqrs$ .... numerum ν, et μ per numerum termini quaeſiti determinatur, poſitoque $\nu = 1, 2, 3, 4, 5$ ... ſucceſſive, fit $N = A, B, C, D, E$.., reſpective) reperitur, facta literarum ν et μ determinatione ſucceſſiva,

p
$^{\nu}A$

(k) *Infinit. Dignit.* §. XXV. *ſ.* 4. *p.* 115, 116.

(l) *Infin. Dign. Append. Tab.* V. *p.* 167.

(m) *Infin. Dign.* §. XXIV. *ſ.* 6. *Exempl.* VI. p. 110.

(n) *Infinit. Dignit.* §. XXVII. *ſ.* 11-13. *p.* 136. 137.

$$\begin{array}{l|l|l|l|l}
\overset{p}{{}^1A}z^1 = p\urcorner 1 & & & & \\
\overset{p}{{}^2A}z^2 = p\urcorner 2 & \overset{qp}{{}^2B}z^2 = pq\urcorner 1 & & & \\
\overset{p}{{}^3A}z^3 = p\urcorner 3 & \overset{qp}{{}^3B}z^3 = pq\urcorner 2 & \overset{rqp}{{}^3C}z^3 = pqr\urcorner 1 & & \\
\overset{p}{{}^4A}z^4 = p\urcorner 4 & \overset{qp}{{}^4B}z^4 = pq\urcorner 3 & \overset{rqp}{{}^4C}z^4 = pqr\urcorner 2 & \overset{s..p}{{}^4D} = pqrs\urcorner 1 & \\
\text{etc} \quad \text{etc} & \text{etc} \quad \text{etc} & \text{etc} \quad \text{etc} & \text{etc} \quad \text{etc} & \text{etc, etc} \\
\overset{p}{{}^nA}z^n = p\urcorner n & \overset{qp}{{}^nB}z^n = pq\urcorner(n-1) & \overset{rqp}{{}^nC}z^n = pqr\urcorner(n-2) & \overset{s..p}{{}^nD} = pqrs\urcorner(n-3) & \text{etc, etc}
\end{array}$$

Similiterque $\overset{t...p}{{}^nE} = pqrst\urcorner(n-4)$; $\overset{u...p}{{}^nF} = pqrstu\urcorner(n-5)$ et ſic porro.

Hinc, quia [V, 2] *Coefficiens termini* $pqrs\ldots\urcorner n = pqrs\ldots\varkappa\, n$, fit

$$\begin{array}{l|l|l|l}
\overset{p}{{}^1A} = p\varkappa 1 & & & \\
\overset{p}{{}^2A} = p\varkappa 2 & \overset{qp}{{}^2B} = pq\varkappa 1 & & \\
\overset{p}{{}^3A} = p\varkappa 3 & \overset{qp}{{}^3B} = pq\varkappa 2 & \overset{rqp}{{}^3C} = pqr\varkappa 1 & \\
\overset{p}{{}^4A} = p\varkappa 4 & \overset{qp}{{}^4B} = pq\varkappa 3 & \overset{rqp}{{}^4C} = pqr\varkappa 2 & \overset{s..p}{{}^4D} = pqrs\varkappa 1 \\
\text{etc} \quad \text{etc} & \text{etc} & \text{etc} & \text{etc} \\
\overset{p}{{}^nA} = p\varkappa n & \overset{qp}{{}^nB} = pq\varkappa(n-1) & \overset{rpq}{{}^nC} = pqr\varkappa(n-2) & \overset{s..p}{{}^nD} = pqrs\varkappa(n-3)
\end{array}$$

et proinde $\overset{t...p}{{}^nE} = pqrst\,\varkappa(n-4)$; $\overset{u...p}{{}^nF} = pqrstu\,\varkappa(n-5)$ etc, etc.

Formula igitur data quaelibet, Elementa combinatoria $\overset{p}{{}^nA}, \overset{qp}{{}^nB}, \overset{rqp}{{}^nC}\ldots$ Productorum a Seriebus diverſis $p, q, r\ldots$, exhibens, ſtatim nunc in aliam poterit converti, quae indicabit, *quo ordine*, *quove modo* Seriei fundamentalis a Producto $pqrs\ldots$ oriundae termini ſinguli $\urcorner$ vel t, aut terminorum coefficientes $\varkappa$ vel c, Seriem datam vel quaeſitam ingrediantur atque inde pendeant, et contra, datis Seriei quaeſitae terminis ſive coefficientibus, per Seriei primitivae terminos ſive coefficientes, indicabit ſimul, quomodo Elementa combinatoria inde, et horum ope Coefficientes ipſi dati, poſſint deduci.

Exemplis, quae hic afferri poſſent (*), ſuperſedebimus, quia eodem modo tractantur, ut ea, quae mox dedimus in antecedentibus. Ceterum expedit plerumque terminos, qui quaeruntur, per Elementa potius combinatoria, quam per terminos ſive Coefficientes Seriei vel Serierum elementarium exprimere, quia brevior eſt fere Elementorum notatio, et multas praeterea habet commoditates, propter Binomiales Coefficientes et alios, qui iunguntur interdum his Elementis, et pariter pendent a Claſſibus *A*, *B*, *C*... per quas determinantur; ut patet ex termino $\Gamma^{(4)}z^4$ Exempli praecedentis Imi.

Coefficientium in Serierum Dignitatibus relationes.

7. Seriei $\alpha z^{\mu} + \beta z^{\mu+\delta} + \gamma z^{\mu+2\delta} + \delta z^{\mu+3\delta} + \varepsilon z^{\mu+4\delta} + \text{etc} = p$, pendet a termino generali (*p*) Dignitas Fxponentis cuiuslibet ν, *integri* et *fracti*, *poſitivi* vel *negativi*, h. e.

$$p^{\nu} = \alpha^{\nu} z^{\nu\mu} + {}^{\nu}\mathfrak{A}\,\mathfrak{a}^{1}\mathrm{A}\,\alpha^{\nu-1} z^{\nu\mu+\delta} + ({}^{\nu}\mathfrak{A}\,\mathfrak{a}^{2}\mathrm{A}\,\alpha^{\nu-1} + {}^{\nu}\mathfrak{B}\,\mathfrak{b}^{2}\mathrm{B}\,\alpha^{\nu-2}) z^{\nu\mu+2\delta}$$
$$+ ({}^{\nu}\mathfrak{A}\,\mathfrak{a}^{3}\mathrm{A}\,\alpha^{\nu-1} + {}^{\nu}\mathfrak{B}\,\mathfrak{b}^{3}\mathrm{B}\,\alpha^{\nu-2} + {}^{\nu}\mathfrak{C}\,\mathfrak{c}^{3}\mathrm{C}\,\alpha^{\nu-3}) z^{\nu\mu+3\delta} \ldots\ldots$$
$$\ldots + ({}^{\nu}\mathfrak{A}\,\mathfrak{a}^{n}\mathrm{A}\,\alpha^{\nu-1} + {}^{\nu}\mathfrak{B}\,\mathfrak{b}^{n}\mathrm{B}\,\alpha^{\nu-2} + {}^{\nu}\mathfrak{C}\,\mathfrak{c}^{n}\mathrm{C}\,\alpha^{\nu-3} \ldots + {}^{\nu}\mathfrak{N}\,\mathfrak{n}^{n} N \alpha^{\nu-n}) z^{\nu\mu+n\delta}$$

ubi Claſſis indefinita *N*, pariter ac $\mathfrak{n}$ et $\mathfrak{N}$, a numeri *n* pendent determinatione ſic ut, poſito $n = 1, 2, 3, 4 \ldots$ ſucceſſive, fiat $N = \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D} \ldots$ et $\mathfrak{n} = \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \ldots$ et ${}^{\nu}\mathfrak{N} = {}^{\nu}\mathfrak{A}, {}^{\nu}\mathfrak{B}, {}^{\nu}\mathfrak{C}, {}^{\nu}\mathfrak{D} \ldots$ reſpective. Coefficientis autem cuiuslibet definiti tot ſunt partes, quot ſunt numeri *n* unitates.

8. Similiter Dignitates exhibentur (*q*) Exponentium ex ordine $1, 2, 3 \ldots \nu$

$$p^{1} = \mathfrak{a}({}^{1}\mathrm{A}\,z^{\mu} + {}^{2}\mathrm{A}\,z^{\mu+\delta} + {}^{3}\mathrm{A}\,z^{\mu+2\delta} + {}^{4}\mathrm{A}\,z^{\mu+3\delta} + \text{etc})$$
$$p^{2} = \mathfrak{b}({}^{2}\mathrm{B}\,z^{2\mu} + {}^{3}\mathrm{B}\,z^{2\mu+\delta} + {}^{4}\mathrm{B}\,z^{2\mu+2\delta} + {}^{5}\mathrm{B}\,z^{2\mu+3\delta} + \text{etc})$$
$$p^{3} = \mathfrak{c}({}^{3}\mathrm{C}\,z^{3\mu} + {}^{4}\mathrm{C}\,z^{3\mu+\delta} + {}^{5}\mathrm{C}\,z^{3\mu+2\delta} + {}^{6}\mathrm{C}\,z^{3\mu+3\delta} + \text{etc})$$
$$\text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc}$$

$$\text{et } p^{\nu} = \mathfrak{n}({}^{\nu}N z^{\nu\mu} + {}^{\nu+1}N z^{\nu\mu+\delta} + {}^{\nu+2}N z^{\nu\mu+2\delta} \ldots + {}^{\nu+n}N z^{\nu\mu+n\delta})$$

ſed pro Exponente tantum integro poſitivo ν, a cuius determinatione ſucceſſiva per numeros 1, 2, 3, 4... definitur etiam Claſſis *N* per A, B, C, D....

9. Poſitis

(*o*) Exempla extant complura *Infin. Dignit.* §. XXVII, inde a ſ. 14.

(*p*) *Infinit. Dignit.* §. XXV, 5, p. 119. ubi occurrit terminus indefinitus $(n+1)$tus Exponentis m.

(*q*) *Infin. Dignit.* §. XXIII, ſ. 7, 8. p. 98. Nempe Serierum forma patet ex p^{ν} in 7.

9. Poſitis itaque Dignitatibus p^{ν} in 7 et 8 aequalibus; id quod pro literae ν valore quolibet integro poſitivo fieri poteſt, producitur comparandis Coefficientibus terminorum homogeneorum ſive eorundem Exponentium:

$$\mathfrak{n}^{\nu} N = \alpha^{\nu}$$
$$\mathfrak{n}^{\nu+1} N = {}^{\nu}\mathfrak{A}\,\mathfrak{a}^{1}\mathrm{A}\,\alpha^{\nu-1}$$
$$\mathfrak{n}^{\nu+2} N = {}^{\nu}\mathfrak{A}\,\mathfrak{a}^{2}\mathrm{A}\,\alpha^{\nu-1} + {}^{\nu}\mathfrak{B}\,\mathfrak{b}^{2}\mathrm{B}\,\alpha^{\nu-2}$$
$$\mathfrak{n}^{\nu+3} N = {}^{\nu}\mathfrak{A}\,\mathfrak{a}^{3}\mathrm{A}\,\alpha^{\nu-1} + {}^{\nu}\mathfrak{B}\,\mathfrak{b}^{3}\mathrm{B}\,\alpha^{\nu-2} + {}^{\nu}\mathfrak{C}\,\mathfrak{c}^{3}\mathrm{C}\,\alpha^{\nu-3}$$

etc etc etc etc

$$\mathfrak{n}^{\nu+n} N = {}^{\nu}\mathfrak{A}\,\mathfrak{a}^{n}\mathrm{A}\,\alpha^{\nu-1} + {}^{\nu}\mathfrak{B}\,\mathfrak{b}^{n}\mathrm{B}\,\alpha^{\nu-2} + {}^{\nu}\mathfrak{C}\,\mathfrak{c}^{n}\mathrm{C}\,\alpha^{\nu-3} \ldots + {}^{\nu}\mathfrak{N}\,\mathfrak{n}^{n} N\,\alpha^{\nu-n}$$

10. Hinc fit, ſubſtituendo pro ν numeros 1, 2, 3, 4, 5 . . . ſucceſſive, in aequationem *primam*, *ſecundam*, *tertiam*, *quartam* . . . . *ntam*

pro $\nu = 1$	pro $\nu = 2$	pro $\nu = 3$
$\mathfrak{a}^{1}\mathrm{A} = \alpha$	$\mathfrak{b}^{2}\mathrm{B} = \alpha^{2}$	$\mathfrak{c}^{3}\mathrm{C} = \alpha^{3}$
$\mathfrak{a}^{2}\mathrm{A} = {}^{1}\mathfrak{A}\,\mathfrak{a}^{1}\mathrm{A}$	$\mathfrak{b}^{3}\mathrm{B} = {}^{2}\mathfrak{A}\,\mathfrak{a}^{1}\mathrm{A}\,\alpha$	$\mathfrak{c}^{4}\mathrm{C} = {}^{3}\mathfrak{A}\,\mathfrak{a}^{1}\mathrm{A}\,\alpha^{2}$
$\mathfrak{a}^{3}\mathrm{A} = {}^{1}\mathfrak{A}\,\mathfrak{a}^{2}\mathrm{A}$	$\mathfrak{b}^{4}\mathrm{B} = {}^{2}\mathfrak{A}\,\mathfrak{a}^{2}\mathrm{A}\,\alpha + {}^{2}\mathfrak{B}\,\mathfrak{b}^{2}\mathrm{B}$	$\mathfrak{c}^{5}\mathrm{C} = {}^{3}\mathfrak{A}\,\mathfrak{a}^{2}\mathrm{A}\,\alpha^{2} + {}^{3}\mathfrak{B}\,\mathfrak{b}^{2}\mathrm{B}\,\alpha$
$\mathfrak{a}^{4}\mathrm{A} = {}^{1}\mathfrak{A}\,\mathfrak{a}^{3}\mathrm{A}$	$\mathfrak{b}^{5}\mathrm{B} = {}^{2}\mathfrak{A}\,\mathfrak{a}^{3}\mathrm{A}\,\alpha + {}^{2}\mathfrak{B}\,\mathfrak{b}^{3}\mathrm{B}$	$\mathfrak{c}^{6}\mathrm{C} = {}^{3}\mathfrak{A}\,\mathfrak{a}^{3}\mathrm{A}\,\alpha^{2} + {}^{3}\mathfrak{B}\,\mathfrak{b}^{3}\mathrm{B}\,\alpha + {}^{3}\mathfrak{C}\,\mathfrak{c}^{3}\mathrm{C}$
etc etc	etc etc	etc etc etc
$\mathfrak{a}^{1+n}\mathrm{A} = {}^{1}\mathfrak{A}\,\mathfrak{a}^{n}\mathrm{A}$	$\mathfrak{b}^{2+n}\mathrm{B} = {}^{2}\mathfrak{A}\,\mathfrak{a}^{n}\mathrm{A}\,\alpha + {}^{2}\mathfrak{B}\,\mathfrak{b}^{n}\mathrm{B}$	$\mathfrak{c}^{3+n}\mathrm{C} = {}^{3}\mathfrak{A}\,\mathfrak{a}^{n}\mathrm{A}\,\alpha^{2} + {}^{3}\mathfrak{B}\,\mathfrak{b}^{n}\mathrm{B}\,\alpha + {}^{3}\mathfrak{C}\,\mathfrak{c}^{n}\mathrm{C}$

ubi *Complexionum* in Claſſibus reſpondent,

a laeva ſigni aequalitatis,

numeris 1, 2, 3, 4, 5 . . . . *ex ordine*,

Coefficientes α, β, γ, δ, ε . . . . *reſpective*,

a dextra autem ſigni aequalitatis

numeris 1, 2, 3, 4, 5 . . . . . *ex ordine*,

Coefficientes β, γ, δ, ε, ζ . . . . . *reſpective*;

nam in Claſſibus a dextra iam apparent Coefficientis α poteſtates.

11. Hinc pro Claſſibus pluribus diverſis *inferioribus*, una poteſt, ſed *ſuperior* Claſſis, ſubſtitui, et vice verſa, pro ſuperiore plures inferiores; quod plurimum prodeſt, etiam oſtendit, quomodo Claſſes ſuperiores ab inferioribus pendeant, atque inde definiri poſſint.

EXEMPLUM. Quod ſi exhibendae proponerentur Claſſis $\mathfrak{g}^{12}G$ ſpecies (quae in Tab. Vta Appendicis ad Infin. Dign. non apparent) ex *inferiorum* Claſſium ſpeciebus; foret (quia G *ſeptima* ordine litera eſt) $\nu = 7$, adeoque $n = 5$ (quia debet eſſe $\nu + n = 12$) et proinde

$$\mathfrak{g}^{12}G = \begin{cases} {}^{7}\mathfrak{A}\mathfrak{a}^{5}A\alpha^{6} + {}^{7}\mathfrak{B}\mathfrak{b}^{5}\beta\alpha^{5} + {}^{7}\mathfrak{C}\mathfrak{c}^{5}C\alpha^{4} + {}^{7}\mathfrak{D}\mathfrak{d}^{5}D\alpha^{3} + {}^{7}\mathfrak{E}\mathfrak{e}^{5}E\alpha^{2} + {}^{7}\mathfrak{F}\mathfrak{f}^{5}F\alpha + {}^{7}\mathfrak{G}\mathfrak{g}^{5}G \\ 7\alpha^{6}\zeta + 2.21\alpha^{5}\beta\varepsilon + 2.21\alpha^{5}\gamma\delta + 3.35\alpha^{4}\beta^{2}\delta + 3.35\alpha^{4}\beta\gamma^{2} + 4.35\alpha^{3}\beta^{3}\gamma + 21\alpha^{2}\beta^{5} \end{cases}$$

Hic ${}^{7}\mathfrak{F}\mathfrak{f}^{5}F\alpha = 0 = {}^{7}\mathfrak{G}\mathfrak{g}^{5}G$, propter $\mathfrak{f}^{5}F = 0 = \mathfrak{g}^{5}G$.

Sed de Signorum relatione hactenus. Reliquum eſt, ut, quos habeant ſigna haec valores in numeris aut Coefficientibus, breviter explicetur.

§. VII.

Elementa combinatoria praecipua explicantur, valoribus eorum in numeris et coefficientibus per Tabulas diſpoſitis.

Ex Tabulis combinatoriis, quarum iam partem dedimus in Appendice ad libellum de Infinitinomii Dignitatibus, paucas tantum hic et non niſi eas transſcribemus, quibus opus erit ad explicanda ſpecimina Logiſticae Serierum, quae mox afferemus. Huc pertinent *Permutationum*, *Complicationum* ac *Variationum Tabulae* ſequentes:

Tabula I. *Permutationes rerum duarum, trium, quatuor, quinque diverſarum.*

1) *Permutationes* $\begin{Bmatrix}1, 2\\ a, b\end{Bmatrix} = \begin{Bmatrix}12; & 21\\ ab; & ba\end{Bmatrix}$

2) *Permutationes* $\begin{Bmatrix}1, 2, 3\\ a, b, c\end{Bmatrix} = \begin{Bmatrix}123, & 132; & 213, & 231; & 312, & 321\\ abc, & acb; & bac, & bca; & cab, & cba\end{Bmatrix}$

3) *Permutationes* $\begin{Bmatrix}1, 2, 3, 4\\ a, b, c, d\end{Bmatrix}$

1234, *abcd*	1423, *adbc*	2314, *bcad*	3124, *cabd*	3412, *cdab*	4213, *dbac*
1243, *abdc*	1432, *adcb*	2341, *bcda*	3142, *cadb*	3421, *cdba*	4231, *dbca*
1324, *acbd*	2134, *bacd*	2413, *bdac*	3214, *cbad*	4123, *dabc*	4312, *dcab*
1342, *acdb*	2143, *badc*	2431, *bdca*	3241, *cbda*	4132, *dacb*	4321, *dcba*

4) *Permutationes* $\begin{Bmatrix}1, 2, 3, 4, 5\\ a, b, c, d, e\end{Bmatrix}$

12345, *abcde*	21345, *bacde*	31245, *cabde*	41235, *dabce*	51234, *eabcd*
12354, *abced*	21354, *baced*	31254, *cabed*	41253, *dabec*	51243, *eabdc*
12435, *abdce*	21435, *badce*	31425, *cadbe*	41325, *dacbe*	51324, *eacbd*
12453, *abdec*	21453, *badec*	31452, *cadeb*	41352, *daceb*	51342, *eacdb*
12534, *abecd*	21534, *baecd*	31524, *caebd*	41523, *daebc*	51423, *eadbc*
12543, *abedc*	21543, *baedc*	31542, *caedb*	41532, *daecb*	51432, *eadcb*
13245, *acbde*	23145, *bcade*	32145, *cbade*	42135, *dbace*	52134, *ebacd*
13254, *acbed*	23154, *bcaed*	32154, *cbaed*	42153, *dbaec*	52143, *ebadc*
13425, *acdbe*	23415, *bcdae*	32415, *cbdae*	42315, *dbcae*	52314, *ebcad*
13452, *acdeb*	23451, *bcdea*	32451, *cbdea*	42351, *dbcea*	52341, *ebcda*
13524, *acebd*	23514, *bcead*	32514, *cbead*	42513, *dbeac*	52413, *ebdac*
13542, *acedb*	23541, *bceda*	32541, *cbeda*	42531, *dbeca*	52431, *ebdca*
14235, *adbce*	24135, *bdace*	34125, *cdabe*	43125, *dcabe*	53124, *ecabd*
14253, *adbec*	24153, *bdaec*	34152, *cdaeb*	43152, *dcaeb*	53142, *ecadb*
14325, *adcbe*	24315, *bdcae*	34215, *cdbae*	43215, *dcbae*	53214, *ecbad*
14352, *adceb*	24351, *bdcea*	34251, *cdbea*	43251, *dcbea*	53241, *ecbda*
14523, *adebc*	24513, *bdeac*	34512, *cdeab*	43512, *dceab*	53412, *ecdab*
14532, *adecb*	24531, *bdeca*	34521, *cdeba*	43521, *dceba*	53421, *ecdba*
15234, *aebcd*	25134, *beacd*	35124, *ceabd*	45123, *deabc*	54123, *edabc*
15243, *aebdc*	25143, *beadc*	35142, *ceadb*	45132, *deacb*	54132, *edacb*
15324, *aecbd*	25314, *becad*	35214, *cebad*	45213, *debac*	54213, *edbac*
15342, *aecdb*	25341, *becda*	35241, *cebda*	45231, *debca*	54231, *edbca*
15423, *aedbc*	25413, *bedac*	35412, *cedab*	45312, *decab*	54312, *edcab*
15432, *aedcb*	25431, *bedca*	35421, *cedba*	45321, *decba*	54321, *edcba*

Tabula II.

Tabula II. *Complexionum Claſſes a numeris definitis* 1—10.

Classis	Complexio	Nr.
^{1}A	1	(1
^{2}A	2	(1
^{2}B	1,1	(1
^{3}A	3	(1
^{3}B	1,2	(1
^{3}C	1,1,1	(1
^{4}A	4	(1
^{4}B	1,3	(1
	2,2	(2
^{4}C	1,1,2	(1
^{4}D	1,1,1,1	(1
^{5}A	5	(1
^{5}B	1,4	(1
	2,3	(2
^{5}C	1,1,3	(1
	1,2,2	(2
^{5}D	1,1,1,2	(1
^{5}E	1,1,1,1,1	(1
^{6}A	6	(1
^{6}B	1,5	(1
	2,4	(2
	3,3	(3
^{6}C	1,1,4	(1
	1,2,3	(2
	2,2,2	(3
^{6}D	1,1,1,3	(1
	1,1,2,2	(2
^{6}E	1,1,1,1,2	(1
^{6}F	1,1,1,1,1,1	(1
^{7}A	7	(1
^{7}B	1,6	(1
	2,5	(2
	3,4	(3
^{7}C	1,1,5	(1
	1,2,4	(2
	1,3,3	(3
	2,2,3	(4
^{7}D	1,1,1,4	(1
	1,1,2,3	(2
	1,2,2,2	(3
^{7}E	1,1,1,1,3	(1
	1,1,1,2,2	(2
^{7}F	1,1,1,1,1,2	(1
^{7}G	1,1,1,1,1,1,1	(1
^{8}A	8	(1
^{8}B	1,7	(1
	2,6	(2
	3,5	(3
	4,4	(4
^{8}C	1,1,6	(1
	1,2,5	(2
	1,3,4	(3
	2,2,4	(4
	2,3,3	(5
^{8}D	1,1,1,5	(1
	1,1,2,4	(2
	1,1,3,3	(3
	1,2,2,3	(4
	2,2,2,2	(5
^{8}E	1,1,1,1,4	(1
	1,1,1,2,3	(2
	1,1,2,2,2	(3
^{8}F	1,1,1,1,1,3	(1
	1,1,1,1,2,2	(2
^{8}G	1,1,1,1,1,1,2	(1
^{8}H	1,1,1,1,1,1,1,1	(1
^{9}A	9	(1
^{9}B	1,8	(1
	2,7	(2
	3,6	(3
	4,5	(4
^{9}C	1,1,7	(1
	1,2,6	(2
	1,3,5	(3
	1,4,4	(4
	2,2,5	(5
	2,3,4	(6
	3,3,3	(7
^{9}D	1,1,1,6	(1
	1,1,2,5	(2
	1,1,3,4	(3
	1,2,2,4	(4
	1,2,3,3	(5
	2,2,2,3	(6
^{9}E	1,1,1,1,5	(1
	1,1,1,2,4	(2
	1,1,1,3,3	(3
	1,1,2,2,3	(4
	1,2,2,2,2	(5
^{9}F	1,1,1,1,1,4	(1
	1,1,1,1,2,3	(2
	1,1,1,2,2,2	(3
^{9}G	1,1,1,1,1,1,3	(1
	1,1,1,1,1,2,2	(2
^{9}H	1,1,1,1,1,1,1,2	(1
^{9}I	1,1,1,1,1,1,1,1,1	(1
^{10}A	10	(1
^{10}B	1,9	(1
	2,8	(2
	3,7	(3
	4,6	(4
	5,5	(5
^{10}C	1,1,8	(1
	1,2,7	(2
	1,3,6	(3
	1,4,5	(4
	2,2,6	(5
	2,3,5	(6
	2,4,4	(7
	3,3,4	(8
^{10}D	1,1,1,7	(1
	1,1,2,6	(2
	1,1,3,5	(3
	1,1,4,4	(4
	1,2,2,5	(5
	1,2,3,4	(6
	1,3,3,3	(7
	2,2,2,4	(8
	2,2,3,3	(9
^{10}E	1,1,1,1,6	(1
	1,1,1,2,5	(2
	1,1,1,3,4	(3
	1,1,2,2,4	(4
	1,1,2,3,3	(5
	1,2,2,2,3	(6
	2,2,2,2,2	(7
^{10}F	1,1,1,1,1,5	(1
	1,1,1,1,2,4	(2
	1,1,1,1,3,3	(3
	1,1,1,2,2,3	(4
	1,1,2,2,2,2	(5
^{10}G	1,1,1,1,1,1,4	(1
	1,1,1,1,1,2,3	(2
	1,1,1,1,2,2,2	(3
^{10}H	1,1,1,1,1,1,1,3	(1
	1,1,1,1,1,1,2,2	(2
^{10}I	1,1,1,1,1,1,1,1,2	(1
^{10}K	1,1,1,1,1,1,1,1,1,1	(1

Hac Tabula exhibentur *Uniones* (1, 2, 3 ... 10) = (^{1}A, ^{2}A ... ^{10}A) et *Combinationes* (1, 2, 3 ... 9) : (^{2}B, ^{3}B ... ^{10}B) et *Conternationes* (1, 2, 3 ... 8) = (^{3}C, ^{4}C ... ^{10}C) et reliquae *Complexiones*, usque ad 1,1,1,1,1,1,1,1,1,1 = ^{10}K, *numerorum* omnes *definitorum* coniunctim : ^{1}A; (^{2}A, ^{2}B); (^{3}A, ^{3}B, ^{3}C) ... ... [illegible] ſingulas numerant formas Claſſium

Tabula III. *Complexionum Claſſes numeri definiti et Species ſingulae, cum numeris transpoſitionum, poſitis pro Tabulae antecedentis*

Numeris 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 *ex ordine*,

Coefficientibus α, β, γ, δ, ε, ζ, η, θ, ι, κ *reſpective*

$\mathfrak{a}^{1}A = \alpha$ (1

$\mathfrak{a}^{2}A = \beta$ (1
$\mathfrak{b}^{2}B = \alpha$ (1

$\mathfrak{a}^{3}A = \gamma$ (1
$\mathfrak{b}^{3}B = 2\alpha\beta$ (1
$\mathfrak{c}^{3}C = \alpha^{3}$ (1

$\mathfrak{a}^{4}A = \delta$ (1
$\mathfrak{b}^{4}B =$ $2\alpha\gamma$ (1, β^{2} (2
$\mathfrak{c}^{4}C = 3\alpha^{2}\beta$ (1
$\mathfrak{d}^{4}D = \alpha^{4}$ (1

$\mathfrak{a}^{5}A = \varepsilon$ (1
$\mathfrak{b}^{5}B =$ $2\alpha\delta$ (1, $2\beta\gamma$ (2
$\mathfrak{c}^{5}C =$ $3\alpha^{2}\gamma$ (1, $3\alpha\beta^{2}$ (2
$\mathfrak{d}^{5}D = 4\alpha^{3}\beta$ (1
$\mathfrak{e}^{5}E = \alpha^{5}$ (1

$\mathfrak{a}^{6}A = \zeta$ (1
$\mathfrak{b}^{6}B =$ $2\alpha\varepsilon$ (1, $2\beta\delta$ (2, γ^{2} (3
$\mathfrak{c}^{6}C =$ $3\alpha^{2}\delta$ (1, $6\alpha\beta\gamma$ (2, β^{3} (3
$\mathfrak{d}^{6}D =$ $4\alpha^{3}\gamma$ (1, $6\alpha^{2}\beta^{2}$ (2
$\mathfrak{e}^{6}E = 5\alpha^{4}\beta$ (1
$\mathfrak{f}^{6}F = \alpha$ (1

$\mathfrak{a}^{7}A = \eta$ (1
$\mathfrak{b}^{7}B =$ $2\alpha\zeta$ (1, $2\beta\varepsilon$ (2, $2\gamma\delta$ (3
$\mathfrak{c}^{7}C =$ $3\alpha^{2}\varepsilon$ (1, $6\alpha\beta\delta$ (2, $3\alpha\gamma^{2}$ (3, $3\beta^{2}\gamma$ (4
$\mathfrak{d}^{7}D =$ $4\alpha^{3}\delta$ (1, $12\alpha^{2}\beta\gamma$ (2, $4\alpha\beta^{3}$ (3
$\mathfrak{e}^{7}E =$ $5\alpha^{4}\gamma$ (1, $10\alpha^{3}\beta^{2}$ (2
$\mathfrak{f}^{7}F = 6\alpha^{5}\beta$ (1
$\mathfrak{g}^{7}G = \alpha^{7}$ (1

$\mathfrak{a}^{8}A = \theta$ (1
$\mathfrak{b}^{8}B =$ $2\alpha\eta$ (1, $2\beta\zeta$ (2, $2\gamma\varepsilon$ (3, δ^{2} (4
$\mathfrak{c}^{8}C =$ $3\alpha^{2}\zeta$ (1, $6\alpha\beta\varepsilon$ (2, $6\alpha\gamma\delta$ (3, $3\beta^{2}\delta$ (4, $3\beta\gamma^{2}$ (5
$\mathfrak{d}^{8}D =$ $4\alpha^{3}\varepsilon$ (1, $12\alpha^{2}\beta\delta$ (2, $6\alpha^{2}\gamma^{2}$ (3, $12\alpha\beta^{2}\gamma$ (4, β^{4} (5
$\mathfrak{e}^{8}E =$ $5\alpha^{4}\delta$ (1, $20\alpha^{3}\beta\gamma$ (2, $10\alpha^{2}\beta^{3}$ (3
$\mathfrak{f}^{8}F =$ $6\alpha^{5}\gamma$ (1, $15\alpha^{4}\beta^{2}$ (2
$\mathfrak{g}^{8}G = 7\alpha^{6}\beta$ (1
$\mathfrak{h}^{8}H = \alpha^{8}$ (1

$\mathfrak{a}^{9}A = \iota$ (1
$\mathfrak{b}^{9}B =$ $2\alpha\theta$ (1, $2\beta\eta$ (2, $2\gamma\zeta$ (3, $2\delta\varepsilon$ (4
$\mathfrak{c}^{9}C =$ $3\alpha^{2}\eta$ (1, $6\alpha\beta\zeta$ (2, $6\alpha\gamma\varepsilon$ (3, $3\alpha\delta^{2}$ (4, $3\beta^{2}\varepsilon$ (5, $6\beta\gamma\delta$ (6, γ^{3} (7
$\mathfrak{d}^{9}D =$ $4\alpha^{3}\zeta$ (1, $12\alpha^{2}\beta\varepsilon$ (2, $12\alpha^{2}\gamma\delta$ (3, $12\alpha\beta^{2}\delta$ (4, $12\alpha\beta\gamma^{2}$ (5, $4\beta^{3}\gamma$ (6
$\mathfrak{e}^{9}E =$ $5\alpha^{4}\varepsilon$ (1, $20\alpha^{3}\beta\delta$ (2, $10\alpha^{3}\gamma^{2}$ (3, $30\alpha^{2}\beta^{2}\gamma$ (4, $5\alpha\beta^{4}$ (5
$\mathfrak{f}^{9}F =$ $6\alpha^{5}\delta$ (1, $30\alpha^{4}\beta\gamma$ (2, $20\alpha^{3}\beta^{3}$ (3
$\mathfrak{g}^{9}G =$ $7\alpha^{6}\gamma$ (1, $21\alpha^{5}\beta^{2}$ (2
$\mathfrak{h}^{9}H = 8\alpha^{7}\beta$ (1
$\mathfrak{i}^{9}I = \alpha^{9}$ (1

$\mathfrak{a}^{10}A = \kappa$ (1
$\mathfrak{b}^{10}B =$ $2\alpha\iota$ (1, $2\beta\theta$ (2, $2\gamma\eta$ (3, $2\delta\zeta$ (4, ε^{2} (5
$\mathfrak{c}^{10}C =$ $3\alpha^{2}\theta$ (1, $6\alpha\beta\eta$ (2, $6\alpha\gamma\zeta$ (3, $6\alpha\delta\varepsilon$ (4, $3\beta^{2}\zeta$ (5, $6\beta\gamma\varepsilon$ (6, $3\beta\delta^{2}$ (7, $3\gamma^{2}\delta$ (8
$\mathfrak{d}^{10}D =$ $4\alpha^{3}\eta$ (1, $12\alpha^{2}\beta\zeta$ (2, $12\alpha^{2}\gamma\varepsilon$ (3, $6\alpha^{2}\delta^{2}$ (4, $12\alpha\beta^{2}$ (5, $24\alpha\beta\gamma\delta$ (6, $4\alpha\gamma^{3}$ (7, $4\beta^{3}\delta$ (8, $6\beta^{2}\gamma^{2}$ (9
$\mathfrak{e}^{10}E =$ $5\alpha^{4}\zeta$ (1, $20\alpha^{3}\beta\varepsilon$ (2, $20\alpha^{3}\gamma\delta$ (3, $30\alpha^{2}\beta^{2}\delta$ (4, $30\alpha^{2}\beta\delta^{2}$ (5, $20\alpha\beta^{3}\gamma$ (6, β^{5} (7
$\mathfrak{f}^{10}F =$ $6\alpha^{5}\varepsilon$ (1, $30\alpha^{4}\beta\delta$ (2, $15\alpha^{4}\gamma^{2}$ (3, $60\alpha^{3}\beta^{2}\gamma$ (4, $15\alpha^{2}\beta^{4}$ (5
$\mathfrak{g}^{10}G =$ $7\alpha^{6}\delta$ (1, $42\alpha^{5}\beta\gamma$ (2, $35\alpha^{4}\beta^{3}$ (3
$\mathfrak{h}^{10}H =$ $8\alpha^{7}\beta$ (1, $28\alpha^{6}\beta^{2}$ (2
$\mathfrak{i}^{10}I = 9\alpha^{8}\beta$ (1
$\mathfrak{k}^{10}K = \alpha^{10}$ (1

Haec Tabula eadem eſt cum praecedente, niſi quod hic literae pro numeris poſitae, et ſingulis ſimul ſpeciebus transpoſitionum numeri ſive coefficientes polynomiales iuncti ſunt. Caeterum ambae Tabulae, maxime haec, uſum habent in Analyſi longe

p p
1) 1 ^1A

p p
1) 2 ^2A
q — qp
1) 1,1 ^2B

p p
1) 3 ^3A
q —
1) 1,2 qp
2) 2,1 ^3B
r — rqp
1) 1,1,1 ^3C

p p
1) 4 ^4A
q —
1) 1,3
2) 2,2 qp
3) 3,1 ^4B
r —
1) 1,1,2 rqp
2) 1,2,1 ^4C
3) 2,1,1
s — s..p
1) 1,1,1,1 ^4D

p p
1) 5 ^5A
q —
1) 1,4
2) 2,3 qp
3) 3,2 ^5B
4) 4,1
r —
1) 1,1,3
2) 1,2,2
3) 1,3,1 rqp
4) 2,1,2 ^5C
5) 2,2,1
6) 3,1,1
s —
1) 1,1,1,2
2) 1,1,2,1 s...p
3) 1,2,1,1 ^5D
4) 2,1,1,1
t — t...p
1) 1,1,1,1,1 ^5E
t,s,r,q,p

p p
1) 6 ^6A
q —
1) 1,5
2) 2,4 qp
3) 3,3 ^6B
4) 4,2
5) 5,1
r —
1) 1,1,4
2) 1,2,3
3) 1,3,2
4) 1,4,1
5) 2,1,3 rqp
6) 2,2,2 ^6C
7) 2,3,1
8) 3,1,2
9) 3,2,1
10) 4,1,1
s —
1) 1,1,1,3
2) 1,1,2,2
3) 1,1,3,1
4) 1,2,1,2
5) 1,2,2,1 s...p
6) 1,3,1,1 ^6D
7) 2,1,1,2
8) 2,1,2,1
9) 2,2,1,1
10) 3,1,1,1
t —
1) 1,1,1,1,2
2) 1,1,1,2,1 t...p
3) 1,1,2,1,1 ^6E
4) 1,2,1,1,1
5) 2,1,1,1,1
u — u...p
1) 1,1,1,1,1,1 ^6F
u,t,s,r,q,p

p p
7 ^7A
q —
1) 1,6
2) 2,5 qp
3) 3,4 ^7B
4) 4,3
5) 5,2
6) 6,1
r —
1) 1,1,5
2) 1,2,4
3) 1,3,3
4) 1,4,2
5) 1,5,1
6) 2,1,4
7) 2,2,3 rqp
8) 2,3,2 ^7C
9) 2,4,1
10) 3,1,3
11) 3,2,2
12) 3,3,1
13) 4,1,2
14) 4,2,1
15) 5,1,1
s —
1) 1,1,1,4
2) 1,1,2,3
3) 1,1,3,2
4) 1,1,4,1
5) 1,2,1,3
6) 1,2,2,2
7) 1,2,3,1
8) 1,3,1,2
9) 1,3,2,1 s...p
10) 1,4,1,1 ^7D
s,r,q,p

s,r,q,p
11) 2,1,1,3
12) 2,1,2,2
13) 2,1,3,1
14) 2,2,1,2
15) 2,2,2,1
16) 2,3,1,1
17) 3,1,1,2
18) 3,1,2,1
19) 3,2,1,1
20) 4,1,1,1
t —
1) 1,1,1,1,3
2) 1,1,1,2,2
3) 1,1,1,3,1
4) 1,1,2,1,2
5) 1,1,2,2,1
6) 1,1,3,1,1
7) 1,2,1,1,2 t...p
8) 1,2,1,2,1 ^7E
9) 1,2,2,1,1
10) 1,3,1,1,1
11) 2,1,1,1,2
12) 2,1,1,2,1
13) 2,1,2,1,1
14) 2,2,1,1,1
15) 3,1,1,1,1
u —
1) 1,1,1,1,1,2
2) 1,1,1,1,2,1 u...p
3) 1,1,1,2,1,1 ^7F
4) 1,1,2,1,1,1
5) 1,2,1,1,1,1
6) 2,1,1,1,1,1
v — v...p
1) 1,1,1,1,1,1,1 ^7G
v,u,t,s,r,q,p

Hac Tabula exhibentur *Variationum Uniones* ($\overset{p}{^1A}$, $\overset{p}{^2A}$ ... $\overset{p}{^7A}$) = $\overset{p}{(1,2,3\ldots)}$ et *Biniones* ($\overset{qp}{^2B}$, $\overset{qp}{^3B}$ ..., $\overset{qp}{^7B}$) = $\overset{p}{(1,2,3\ldots)}$ $\overset{q}{(1,2,3\ldots)}$ et *Terniones* ($\overset{rqp}{^3C}$, $\overset{rqp}{^4C}$ ... $\overset{rqp}{^7C}$) = $\overset{p}{(1,2,3\ldots)}$ $\overset{q}{(1,2,3\ldots)}$ $\overset{r}{(1,2,3\ldots)}$ et ſic porro, usque ad *7tionem* $\overset{v\ldots p}{^7G}$ = $\overset{v\,u\,t\,s\,r\,q\,p}{1,1,1,1,1,1,1}$, numerorum omnes definitorum

p p qp p qp rqp p qp v...p

Tabula V. *Variationes ſimpliciter* (1, 2, 3, 4, 5, 6, 7, 8, 9) *a Seriebus p, q, r, s,* etc

	p	*p*	*p*	*p*	*p*	*p*	*p*	*p*	*p*
$\overset{p}{'A}$ =	1,	2,	3,	4,	5,	6,	7,	8,	9

	qp	*qp*	*qp*	*qp*	*qp*	*qp*	*qp*	*qp*	*qp*
	11,	12,	13,	14,	15,	16,	17,	18,	19
	21,	22,	23,	24,	25,	26,	27,	28,	29
	31,	32,	33,	34,	35,	36,	37,	38,	39
	41,	42,	43,	44,	45,	46,	47,	48,	49
$\overset{qp}{'B}$ =	51,	52,	53,	54,	55,	56,	57,	58,	59
	61,	62,	63,	64,	65,	66,	67,	68,	69
	71,	72,	73,	74,	75,	76,	77,	78,	79
	81,	82,	83,	84,	85,	86,	87,	88,	89
	91,	92,	93,	94,	95,	96,	97,	98,	99
	qp	*qp*	*qp*	*qp*	*qp*	*qp*	*qp*	*qp*	*qp*

	rqp	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*
	111,	112,	113,	114,	115,	116,	117,	118,	119,	121 ....,	199
	211,	212,	213,	214,	215,	216,	217,	218,	219,	221	299
	311,	312,	313,	314,	315,	316,	317,	318,	319,	321	399
	411,	412,	413,	414,	415,	416,	417,	418,	419,	421	499
$\overset{rqp}{'C}$ =	511,	512,	513,	514,	515,	516,	517,	518,	519,	521	599
	611,	612,	613,	614,	615,	616,	617,	618,	619,	621	699
	711,	712,	713,	714,	715,	716,	717,	718,	719,	721	799
	811,	812,	813,	814,	815,	816,	817,	818,	819,	821	899
	911,	912,	913,	914,	915,	916,	917,	918,	919,	921	999
	rqp	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*	*rqp*

	srqp	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*
	1111,	1112,	1113,	1114,	1115,	1116,	1117,	1118,	1119,	1121	1999
	2111,	2112,	2113,	2114,	2115,	2116,	2117,	2118,	2119,	2121	2999
	3111,	3112,	3113,	3114,	3115,	3116,	3117,	3118,	3119,	3121	3999
	4111,	4112,	4113,	4114,	4115,	4116,	4117,	4118,	4119,	4121	4999
$\overset{srqp}{'D}$ =	5111,	5112,	5113,	5114,	5115,	5116,	5117,	5118,	5119,	5121	5999
	6111,	6112,	6113,	6114,	6115,	6116,	6117,	6118,	6119,	6121	6999
	7111,	7112,	7113,	7114,	7115,	7116,	7117,	7118,	7119,	7121	7999
	8111,	8112,	8113,	8114,	8115,	8116,	8117,	8118,	8119,	8121	8999
	9111,	9112,	9113,	9114,	9115,	9116,	9117,	9118,	9119,	9121	9999
	srqp	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*	*srqp*

Similiter determinantur $\overset{t..p}{'E}$, $\overset{u..p}{'F}$, $\overset{v..p}{'G}$, cum reliquis Claſſibus Variationum ſimpliciter. Additae ſunt Serierum notae *p*, *q*, *r*, *s* . . . , nam plerumque Claſſes hae diverſarum Serierum terminos exponunt. Interdum tamen ad unam pertinent Seriem *p* Claſſium $\overset{p}{'A}$, $\overset{p}{'B}$, $\overset{p}{'C}$ etc ſpecies ac numeri ſinguli, cuius exponunt partium

§. VIII.

Logiſticae Serierum ſpecimina, Tabulis ſequentibus duabus propoſita, breviter explicantur.

1. Logiſticae Serierum ſpecimina, quae ad calcem libri exhibentur, Tabularum duarum priorum [§. IV, p. XXVII] *Multiplicationis* ſcilicet et *Diviſionis*, particulam conſtituunt. Series hic tractantur ſuo quaeque in genere maxime univerſales: $a+b+c+d+e+etc$ terminorum pro lubitu progredientium; et $a z^{\mu} + b z^{\mu+\delta} + c z^{\mu+2\delta} +$ etc per poteſtates literae z, pro Exponentium μ et δ valore quocunque (poſitivo vel negativo, integro vel fracto) arithmetice vel aſcendentium, vel deſcendentium; et denique $a z^{a} + b z^{b} + c z^{c} + d z^{d} +$ etc pro valoribus quibuscunque Coefficientium a, b, c, .. et Exponentium $a, b, c \ldots$ homonymorum (r) quidem, caeteroquin longiſſime diverſorum. Generaliſſima vulgo exiſtimatur Series $a+b+c+d+$ etc, quae reliquas ambas continet, et, facta ſubſtitutione, producit: ſed longe tamen ſuperat, propter uſum inprimis, quem habet in Analyſi, frequentiſſimum, Series $a z^{\mu} + b z^{\mu+\delta} + c z^{\mu+2\delta} +$ etc, et quia eadem poteſt, poſito $z = 1$, ad Seriem $a+b+c+d+$ etc deduci, unde rurſus pendet Series $a z^{a} + b z^{b} + c z^{c} + d z^{d} +$ etc per ſubſtitutionem. Quapropter Seriei $a z^{\mu} + b z^{\mu+\delta} + c z^{\mu+2\delta}$ etc et ſimilium Operationes expoſuimus inprimis accurate, cum terminis qui hinc pendent, generalibus et ſpecialibus, per formulas analytico-combinatorias expreſſis, quibus continetur progreſſus Lex manifeſta, et quae facile poſſunt in terminos ſimpliciſſimos converti. Expoſitae autem apparent Propoſitiones omnes ſic, ut nemini poſſint, qui ſignorum recte teneat notationes, de quibus egimus §. V, obſcurae eſſe. *Complexionum* hic aut *Variationum ſimpliciter* $({}'A,\ {}'B,\ {}'C \ldots {}'\overset{p}{A},\ {}'\overset{qp}{B},\ {}'\overset{rqp}{C} \ldots)$ ſpecies ſingulae eo modo progrediuntur, quo numeri creſcunt, ex datis elementis compoſiti; reliquae autem $({}^{n}A,\ {}^{n}B,\ {}^{n}C \ldots {}^{n}\overset{p}{A},\ {}^{n}\overset{qp}{B},\ {}^{n}\overset{rqp}{C} \ldots)$, *propoſiti* ſcilicet *numeri n*, quo numeri creſcunt, quorum ſimul elementa addendo efficiunt ſummam *n*. Numeri in Complexionibus aut Variationibus ſinguli quorſum pertineant, h. e. quos terminos ſive terminorum Coefficientes deſignent, indicavimus,

(r) Scilicet homonymia haec aptiſſime exprimit *Operationum* ex pluribus terminis in Coefficientibus et Exponentibus, *relationem*, dum terminorum quaeſitorum Exponentes, ex datorum Exponentibus particularibus, eodem modo *adduntur* et *ſubtrahuntur*, quo Coefficientes homonymi in his terminis *multiplicantur* et *dividuntur*, reſpective.

vimus, in *Tabula prima*, Numeros ipſos in vertice terminorum ſive Coefficientium ponendo; in *Tabula altera*, adiecto Indiculo, qui numerorum cum literis relationem exponit. Ubi Numeri, ut in Tabula prima, plurium Serierum p, q, r, s . . . terminis ſive Coefficientibus reſpondent, Series ipſae in vertice literae, quae Claſſem exponit, indicantur, ut ${}^{1}\overset{qp}{B}$, ${}^{n+2}\overset{rqp}{C}$ etc. quibus deinde in Tabula, qua Elementorum propoſitorum valores explicantur, numeri reſpondent, per verticales Series . . . $s\,r\,q\,p$ diſpoſiti.

2. *Tabula* I. *Serierum in Series Multiplicatio.*

Explicatione ulteriori opus non habet. Productorum termini a *duobus* et *tribus* Seriebus, formae triplicis, de qua modo diximus, Prop. 1-6 explicantur. Propoſitio 7 Productum exhibet a Seriebus *quatuor*, ſed formae tantummodo $a z^{n} + b z^{n+d}$ etc. ſcilicet ut brevitati conſuleremus, et quia reliquae formae ab hac [1] pendent et facile inde deduci poſſunt. Propoſ. 8 et 9 ſunt maxime generales et latiſſime patentes.

Tabulae huiusmodi uſum et applicationem ad caſus ſpeciales, oſtendunt Exempla complura, quae alibi (^{s}) expoſui. Sed eadem poſſunt brevius et commodius abſolvi, per Tabulas, quas nunc proponimus, in quibus Series generales ($\alpha z^{m} + \beta z^{m+d} + \gamma z^{m+2d}$ . . . .) pro ſpecialioribus ($\alpha z^{1} + \beta z^{2} + \gamma z^{3}$ . . .) adhibentur.

3. EXEMPLUM I. *Propoſitis factoribus tribus p, q et r*

$$\overset{(p)}{(A z^{2} + B z^{4} + C z^{6} \ldots)}\ \overset{(q)}{(a z^{2} + b z^{4} + c z^{6} \ldots)}\ \overset{(r)}{(\mathfrak{A} z^{2} + \mathfrak{B} z^{4} + \mathfrak{C} z^{6} \ldots)} = rqp$$

quorum literae z poteſtates ſecundum Exponentes pares 2, 4, 6, 10 . . . progrediuntur, quaeritur producti, ſive Seriei inde provenientis, terminus $\Gamma^{(12)} z^{12}$, *poteſtatem exhibens* z^{12}, *h. e. quaeritur termini huius coefficiens* $\Gamma^{(12)}$ *per datos factorum p, q et r coefficientes* (^{t}).

Facta

(s) *Infinit. Dignit.* §. XXVI. *ſ.* 14.

(t) Idem Exemplum extat l. c. p. 138, 139. ſed ſolutio ibi non poterat tam expedite abſolvi, quam fieri potuit h. l. Serierum ope generaliorum. Idem valet, ubi Exponentes ſecundum numeros impares 1, 3, 5 . . . aut alios progrediuntur.

Facta Comparatione Serierum hic datarum p, q et r cum generalioribus Prop. 5 Tab. I propoſitis, apparet, eſſe $\mu = \nu = \pi = \delta = 2$, adeoque $\Pi = 6$, et $z^{12} = z^{\Pi + 3\delta}$, terminum autem fore quaeſitum, ex Tabula

$$\Gamma^{(12)} z^{12} = \left\{ \begin{array}{l} \mathfrak{A}aD + \mathfrak{A}bC + \mathfrak{A}cB + \mathfrak{A}dA + \mathfrak{B}aC \\ + \mathfrak{B}bB + \mathfrak{B}cA + \mathfrak{C}aB + \mathfrak{C}bA + \mathfrak{D}aA \end{array} \right\} z^{12},$$

4. Quod ſi eiusdem Producti quaereretur terminus $\Gamma^{(7)} z^7$, pariter foret $\delta = 2$, $\Pi = 6$, adeoque $z^7 = z^{\Pi + \frac{1}{2}\delta}$; cuiusmodi Dignitas literae z cum inter terminos Propoſ. 5tae non occurrat, indicio hoc eſt, eſſe Coefficientem $\Gamma^{(7)} = 0$, ipſumque adeo terminum quaeſitum *Nihilo* aequari.

5. EXEMPLUM II. *Producti huiusmodi*

$$\overset{(p)}{(Az^{-2} + Bz^{-1} \ldots)} \; \overset{(q)}{(az^3 + bz^4 \ldots)} \; \overset{(r)}{(\mathfrak{A} + \mathfrak{B}z^1 \ldots)} \; \overset{(s)}{(az^4 + bz^5 \ldots)} = srqp$$

poſita ubique Exponentium differentia = 1, terminos quoslibet per terminos ſimplices exhibere.

Nempe hic ſunt Propoſ. 7 Exponentes $\mu = -2$; $\nu = 3$; $\pi = 0$; $\sigma = 4$, adeoque $\Sigma = 5$, et $\delta = 1$

$$\text{Ergo } srqp \, \urcorner n = {}^{n+3}\overset{srqp}{D} z^{\Sigma + (n-1)} = {}^{n+3}\overset{srqp}{D} z^{n+4}$$

et $srqp = a\mathfrak{A}aAz^5 + (a\mathfrak{A}aB + a\mathfrak{A}bA + a\mathfrak{B}aA + b\mathfrak{A}aA)z^6 +$ etc, ut in Tabula. Nam Serierum $p, q, r, s \ldots$ *Producta* $\ldots srqp$, quoad *Coefficientes*, ut apparet, eadem ſunt perpetuo, et ſolum differunt literae z poteſtatibus. Idem valet de *Quotientibus* Tabula II. propoſitis.

6. Serierum, quarum non ſolum Exponentes terminorum primorum, ſed etiam, quae his iunguntur, pro reliquis terminis, differentiae D, Δ, $\delta \ldots$ discrepant, huiusmodi ſerierum Producta, Quotientes etc, pendent a Serierum $az^a + bz^b + cz^c +$ etc tractatione. Series formae: $az^s + by^r + cx^q + dw^p +$ etc repraeſentantur per $a + b + c + d +$ etc.

7. *Tabula* II. *Serierum per Series Diviſio.*

Etiam haec Tabula ſatis clara eſt per ſe. Propoſitione Ima Quotientis $\dfrac{1}{1 - \alpha - \beta - \gamma - \delta - \text{etc}}$ termini explicantur, per formulam (P) quae terminos per *Complexiones ſimpliciter*, $'A$, $'B$, $'C$ etc h. e. per numeros naturali ordine creſcentes disponit, itemque per formulam (Q) h. e. per *Complexiones numerorum definitorum* ^1A, $^2A \ldots {}^2B$, $^3B \ldots {}^3C$ etc; unde apparet

paret, huius quotientis terminos eosdem plane eſſe cum *terminis combinatoriis* Tabulae p. LIX. propoſitae, eosque cum *terminis ſummae omnium ex ordine Dignitatum* Exponentium 1, 2, 3, 4 . . . m, Seriei $\alpha + \beta + \gamma + \delta$ etc coincidere; h. e. terminos Quotientis $\frac{1}{1-\alpha-\beta\ldots}$ ſive Dignitatis $(1-\alpha-\beta\ldots)^{-1}$ eſſe vere *fundamentales*, et infinitorum inde Problematum ſolutiones pendere.

8. In termino generali Propoſ. 3, literae a, b . . . . $\overset{-2}{n}$, $\overset{-1}{n}$, $\overset{0}{n}$, *Coefficientes* ſignificant 1*mum*, 2*dum* . . . . (*n*-2) *tum*, (*n*-1) *tum*, *ntum*. Uſus autem Coefficientium, qui a *diſtantiae exponentibus* afficiuntur [V, 8.], plurimas habet opportunitates, propter Characteriſticae, in literis pariter ac numeris, homonymiam, et quia *quilibet* Coefficiens poteſt per *quemlibet* exprimi, *definite* et *indefinite*. Namque ubi ponitur, ab uno aliquo Coefficientium a, b, c, d . . . . *quocunque* incipiendo,

Terminus ſive Coefficiens indefinitus *n*tus = n, et *m*tus = m, formulae inde fluunt canonicae, terminorum identicorum,

$$\overset{-n}{n} = \overset{-m}{m} \quad \text{et} \quad \overset{+n}{m} = \overset{+m}{n}$$

$$\text{Hinc}\ \overset{0}{n} = \overset{n-m}{m} \quad \text{et} \quad \overset{0}{m} = \overset{m-n}{n}$$

$$\text{Ergo}\ \overset{\pm r}{n} = \overset{n\pm r-m}{m} \quad \text{et} \quad \overset{\pm r}{m} = \overset{m\pm r-n}{n}$$

Coefficientes ſcilicet *indefiniti*, *n*tus et (*n* ± *r*) tus, per *indefinitum m*tum expreſſi, et viciſſim.

9. Quod ſi itaque numeretur a *primus*, b *ſecundus*, c *tertius Coefficiens*, et ſic porro, ponatur autem

n ſive *m* = 1, 2, 3, 4, 5 . . . ſucceſſive,

fit n ſive m = a, b, c, d, e . . . reſpective (*)

$$\text{et}\ \overset{\pm r}{n} = \overset{n\pm r-1}{a} = \overset{n\pm r-2}{b} = \overset{n\pm r-3}{c} = \overset{n\pm r-4}{d} = \overset{n\pm r-5}{e} = \text{etc}$$

unde

(*) Quodſi autem numerentur p *primus*, q *ſecundus*, r *tertius* etc *Coefficiens*, literarum *n* vel *m* valoribus 1, 2, 3, 4 . . . ſucceſſive, Coefficientes ex ordine determinantur p, q, r, s . . . reſpective, ut ante a, b, c, d . .; fit autem $\overset{-14}{a} = \overset{-15}{p} = q \ldots$; $\overset{-13}{b} = \overset{-14}{p} = q$ = etc,

p p
1) 1 ${}^{1}A$

p p
1) 2 ${}^{2}A$
q — qp
1) 1,1 ${}^{2}B$

p p
1) 3 ${}^{3}A$
q — qp
1) 1,2
2) 2,1 ${}^{3}B$
r — rqp
1) 1,1,1 ${}^{3}C$

p p
1) 4 ${}^{4}A$
q — qp
1) 1,3
2) 2,2
3) 3,1 ${}^{4}B$
r — rqp
1) 1,1,2
2) 1,2,1
3) 2,1,1 ${}^{4}C$
s — s..p
1) 1,1,1,1 ${}^{4}D$

p p
1) 5 ${}^{5}A$
q — qp
1) 1,4
2) 2,3
3) 3,2
4) 4,1 ${}^{5}B$
r — rqp
1) 1,1,3
2) 1,2,2
3) 1,3,1
4) 2,1,2
5) 2,2,1
6) 3,1,1 ${}^{5}C$
s — s...p
1) 1,1,1,2
2) 1,1,2,1
3) 1,2,1,1
4) 2,1,1,1 ${}^{5}D$
t — t...p
1) 1,1,1,1,1 ${}^{5}E$
t,s,r,q,p

p p
1) 6 ${}^{6}A$
q — qp
1) 1,5
2) 2,4
3) 3,3
4) 4,2
5) 5,1 ${}^{6}B$
r — rqp
1) 1,1,4
2) 1,2,3
3) 1,3,2
4) 1,4,1
5) 2,1,3
6) 2,2,2
7) 2,3,1
8) 3,1,2
9) 3,2,1
10) 4,1,1 ${}^{6}C$
s — s...p
1) 1,1,1,3
2) 1,1,2,2
3) 1,1,3,1
4) 1,2,1,2
5) 1,2,2,1
6) 1,3,1,1
7) 2,1,1,2
8) 2,1,2,1
9) 2,2,1,1
10) 3,1,1,1 ${}^{6}D$
t — t...p
1) 1,1,1,1,2
2) 1,1,1,2,1
3) 1,1,2,1,1
4) 1,2,1,1,1
5) 2,1,1,1,1 ${}^{6}E$
u — u...p
1) 1,1,1,1,1,1 ${}^{6}F$
u,t,s,r,q,p

p p
7 ${}^{7}A$
q — qp
1) 1,6
2) 2,5
3) 3,4
4) 4,3
5) 5,2
6) 6,1 ${}^{7}B$
r — rqp
1) 1,1,5
2) 1,2,4
3) 1,3,3
4) 1,4,2
5) 1,5,1
6) 2,1,4
7) 2,2,3
8) 2,3,2
9) 2,4,1
10) 3,1,3
11) 3,2,2
12) 3,3,1
13) 4,1,2
14) 4,2,1
15) 5,1,1 ${}^{7}C$
s — s...p
1) 1,1,1,4
2) 1,1,2,3
3) 1,1,3,2
4) 1,1,4,1
5) 1,2,1,3
6) 1,2,2,2
7) 1,2,3,1
8) 1,3,1,2
9) 1,3,2,1
10) 1,4,1,1 ${}^{7}D$
s,r,q,p

s,r,q,p
11) 2,1,1,3
12) 2,1,2,2
13) 2,1,3,1
14) 2,2,1,2
15) 2,2,2,1
16) 2,3,1,1
17) 3,1,1,2
18) 3,1,2,1
19) 3,2,1,1
20) 4,1,1,1
t — t...p
1) 1,1,1,1,3
2) 1,1,1,2,2
3) 1,1,1,3,1
4) 1,1,2,1,2
5) 1,1,2,2,1
6) 1,1,3,1,1
7) 1,2,1,1,2
8) 1,2,1,2,1
9) 1,2,2,1,1
10) 1,3,1,1,1
11) 2,1,1,1,2
12) 2,1,1,2,1
13) 2,1,2,1,1
14) 2,2,1,1,1
15) 3,1,1,1,1 ${}^{7}E$
u — u...p
1) 1,1,1,1,1,2
2) 1,1,1,1,2,1
3) 1,1,1,2,1,1
4) 1,1,2,1,1,1
5) 1,2,1,1,1,1
6) 2,1,1,1,1,1 ${}^{7}F$
v — v...p
1) 1,1,1,1,1,1,1 ${}^{7}G$
v,u,t,s,r,q,p

Hac Tabula exhibentur *Variationum Uniones* ($\overset{p}{{}^{1}A}$, $\overset{p}{{}^{2}A}$… $\overset{p}{{}^{7}A}$) = $\overset{p}{(1, 2, 3\ldots)}$ et *Biniones* ($\overset{qp}{{}^{2}B}$, $\overset{qp}{{}^{3}B}$…$\overset{qp}{{}^{7}B}$) = $\overset{p}{(1, 2, 3\ldots)}\,\overset{q}{(1, 2, 3\ldots)}$ et *Terniones* ($\overset{rqp}{{}^{3}C}$, $\overset{rqp}{{}^{4}C}$…$\overset{rqp}{{}^{7}C}$) = $\overset{p}{(1, 2, 3\ldots)}\,\overset{q}{(1, 2, 3\ldots)}\,\overset{r}{(1, 2, 3\ldots)}$ et ſic porro, usque ad *7tionem* $\overset{v\ldots p}{{}^{7}G}$ = $\overset{v\,u\,t\,s\,r\,q\,p}{1, 1, 1, 1, 1, 1, 1}$, numerorum omnes definitorum

LOGISTICAE SERIERUM
FORMULIS ANALYTICO - COMBINATORIIS
PER TABULAS EXHIBENDAE
SPEÇIMINA.

TABULA I.

SERIERUM IN SERIES MULTIPLICATIO.

Hac Tabula continentur

I. Canones generales Productorum

a Seriebus *duabus p* et *q*,	A) Propoſ. 1, 2, 3.
a Seriebus *tribus p, q* et *r*	B) Propoſ. 4, 5, 6.
a Seriebus *quatuor p, q, r*, et *s*	C) Propoſ. 7.
a Seriebus *pluribus quotcunque*	D) Propoſ. 8.

II. Canon Summae Productorum a Seriebus *pluribas diverſis quotcunque* — E) Propoſ. 9.

Numeri in vertice terminorum ſive *Coefficientium* (*Propoſ.* 1-7) *poſiti ſignificant*, *Variationum in Claſſibus* $^{m}\overset{qp}{B}$, $^{n}\overset{rqp}{C}$, $^{r}\overset{srqp}{D}$, ... *Tab.* IV. *p.* LX.

numeris ſingulis 1, 2, 3, 4, 5, 6, 7, 8, 9 ... *ex ordine*
reſpondere Coefficientes, A, B, C, D, E, F, G, H, I ... Seriei p
itemque Coefficientes a, b, c, d, e, f, g, h, i ... Seriei q
et Coefficientes 𝔄, 𝔅, ℭ, 𝔇, 𝔈, 𝔉, 𝔊, ℌ, ℑ ... Seriei r
et Coefficientes 𝔞, 𝔟, 𝔠, 𝔡, 𝔢, 𝔣, 𝔤, 𝔥, 𝔦 ... Seriei s

ſingulos reſpective; ſimiliterque *plures* etiam Coefficientes *plurium* Serierum t, u, v, w, etc.

TABULA I.

SERIERUM IN SERIES MULTIPLICATIO.

A. Productum a Seriebus duabus.

$$\text{1 Sit}\left\{\begin{array}{l} \overset{1}{A}+\overset{2}{B}+\overset{3}{C}+\overset{4}{D}+\overset{5}{E}+\overset{6}{F}+\overset{7}{G}+\overset{8}{H}+\overset{9}{I}+\text{etc} = p \\ a+b+c+d+e+f+g+h+i+\text{etc} = q \end{array}\right\}$$

Erit Productum $qp = {}^{\prime}\overset{qp}{B}$; Ergo [Tab. V. p. LXI.]

$$qp = \left\{\begin{array}{l} aA + aB + aC + aD + aE + aF + aG + aH + aI + \text{etc} \\ + bA + bB + bC + bD + bE + bF + bG + bH + bI + \text{etc} \\ + cA + cB + cC + cD + cE + cF + cG + cH + cI + \text{etc} \\ + dA + dB + dC + dD + dE + dF + dG + dH + dI + \text{etc} \\ + eA + eB + eC + eD + eE + eF + eG + eH + eI + \text{etc} \\ + fA + fB + fC + fD + fE + fF + fG + fH + fI + \text{etc} \\ + gA + gB + gC + gD + gE + gF + gG + gH + gI + \text{etc} \\ + hA + hB + hC + hD + hE + hF + hG + hH + hI + \text{etc} \\ + iA + iB + iC + iD + iE + iF + iG + iH + iI + \text{etc} \\ + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} \end{array}\right.$$

$$\text{2 Sit}\left\{\begin{array}{l} \overset{1}{A}z^{\mu} + \overset{2}{B}z^{\mu+\delta} + \overset{3}{C}z^{\mu+2\delta} + \overset{4}{D}z^{\mu+3\delta} + \overset{5}{E}z^{\mu+4\delta} + \text{etc} = p \\ az^{\nu} + bz^{\nu+\delta} + cz^{\nu+2\delta} + dz^{\nu+3\delta} + ez^{\nu+4\delta} + \text{etc} = q \end{array}\right\}$$

Erit Producti qp Terminus $n$tus generalis

$$qp\rceil n = {}^{n+1}\overset{qp}{B}z^{\mu+\nu+(n-1)\delta}$$

Ergo

$$qp = [{}^{2}\overset{qp}{B} + {}^{3}\overset{qp}{B}z^{\delta} + {}^{4}\overset{qp}{B}z^{2\delta} + {}^{5}\overset{qp}{B}z^{3\delta} + {}^{6}\overset{qp}{B}z^{4\delta} + {}^{7}\overset{qp}{B}z^{5\delta} \ldots\ldots + {}^{n+1}\overset{qp}{B}z^{(n-1)\delta}]\, z^{\mu+\nu}$$

unde, poſito $\mu+\nu = N$, fit [Tab. IV. p. LX.]

TABULA I. SERIERUM IN SERIES MULTIPLICATIO.

$$qp = \left\{\begin{array}{l} aAz^{N} \\ +[aB+bA]z^{N+\delta} \\ +[aC+bB+cA]z^{N+2\delta} \\ +[aD+bC+cB+dA]z^{N+3\delta} \\ +[aE+bD+cC+dB+eA]z^{N+4\delta} \\ +[aF+bE+cD+dC+eB+fA]z^{N+5\delta} \\ +[aG+bF+cE+dD+eC+fB+gA]z^{N+6\delta} \\ +[aH+bG+cF+dE+eD+fC+gB+hA]z^{N+7\delta} \\ +[aI+bH+cG+dF+eE+fD+gC+hB+iA]z^{N+8\delta} \\ +[aK+bI+cH+dG+eF+fE+gD+hC+iB+kA]z^{N+9\delta} \\ + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc} + \text{etc.} \end{array}\right.$$

3 Sit

$$\left\{\begin{array}{l} \overset{1}{A}z^{A} + \overset{2}{B}z^{B} + \overset{3}{C}z^{C} + \overset{4}{D}z^{D} + \overset{5}{E}z^{E} + \overset{6}{F}z^{F} + \text{etc} = p \\ az^{a} + bz^{b} + cz^{c} + dz^{d} + ez^{e} + fz^{f} + \text{etc} = q \end{array}\right\}$$

Erit Producti qp Terminus $n$tus generalis

$$qp\rceil n = [{}^{n+1}\overset{qp}{B}1 + {}^{n+1}\overset{qp}{B}2 + {}^{n+1}\overset{qp}{B}3 + {}^{n+1}\overset{qp}{B}4 \ldots\ldots {}^{n+1}\overset{qp}{B}n]z^{*}$$

Ergo

$$qp = {}^{2}\overset{qp}{B}1.z^{*} + ({}^{3}\overset{qp}{B}1 + {}^{3}\overset{qp}{B}2)z^{*} + ({}^{4}\overset{qp}{B}1 + {}^{4}\overset{qp}{B}2 + {}^{4}\overset{qp}{B}3)z^{*} + \text{etc}$$

Et

$$qp = \left\{\begin{array}{l} aAz^{a+A} \\ +aBz^{a+B} + bAz^{b+A} \\ +aCz^{a+C} + bBz^{b+B} + cAz^{c+A} \\ +aDz^{a+D} + bCz^{b+C} + cBz^{c} + + dAz^{d+A} \\ +aEz^{a+E} + bDz^{b+D} + cCz^{c+C} + dBz^{d+B} + eAz^{e+A} \\ +aFz^{a+F} + bEz^{b+E} + cDz^{c+D} + dCz^{d+C} + eBz^{e+B} + fAz^{f+A} \\ +aGz^{a+G} + bFz^{b+F} + cEz^{c+E} + dDz^{d+D} + eCz^{e+C} + fBz^{f+B} + gAz^{g+A} \\ \quad \text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc} \end{array}\right.$$

Nempe Serierum p et q *Exponentes* A; B, C . . . a, b, c . . . eodem modo in Producti qp terminis ſingulis *adduntur*, quo *Coefficientes* A, B, C . . . a, b, c . . . in his terminis *multiplicantur* [V, 18]

TABU-

TABULA I. SERIERUM IN SERIES MULTIPLICATIO.

B. Productum a Seriebus tribus.

$$4\ \text{Sit}\ \left\{\begin{array}{l} \overset{1}{A}+\overset{2}{B}+\overset{3}{C}+\overset{4}{D}+\overset{5}{E}+\overset{6}{F}+\overset{7}{G}+\overset{8}{H}+\overset{9}{I}+\text{etc}=p \\ a+b+c+d+e+f+g+h+i+\text{etc}=q \\ \mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}+\mathfrak{E}+\mathfrak{F}+\mathfrak{G}+\mathfrak{H}+\mathfrak{I}+\text{etc}=r \end{array}\right\}$$

Erit Productum $rqp = {}'\overset{rqp}{C};$ Ergo [Tab. V. p. LXI.]

$$rqp=\left\{\begin{array}{l}
+\mathfrak{A}aA+\mathfrak{A}aB+\mathfrak{A}aC+\mathfrak{A}aD+\mathfrak{A}aE+\mathfrak{A}aF+\mathfrak{A}aG+\mathfrak{A}aH+\mathfrak{A}aI+\ldots+\mathfrak{A}iI+\text{etc} \\
+\mathfrak{B}aA+\mathfrak{B}aB+\mathfrak{B}aC+\mathfrak{B}aD+\mathfrak{B}aE+\mathfrak{B}aF+\mathfrak{B}aG+\mathfrak{B}aH+\mathfrak{B}aI+\ldots+\mathfrak{B}iI+\text{etc} \\
+\mathfrak{C}aA+\mathfrak{C}aB+\mathfrak{C}aC+\mathfrak{C}aD+\mathfrak{C}aE+\mathfrak{C}aF+\mathfrak{C}aG+\mathfrak{C}aH+\mathfrak{C}aI+\ldots+\mathfrak{C}iI+\text{etc} \\
+\mathfrak{D}aA+\mathfrak{D}aB+\mathfrak{D}aC+\mathfrak{D}aD+\mathfrak{D}aE+\mathfrak{D}aF+\mathfrak{D}aG+\mathfrak{D}aH+\mathfrak{D}aI+\ldots+\mathfrak{D}iI+\text{etc} \\
+\mathfrak{E}aA+\mathfrak{E}aB+\mathfrak{E}aC+\mathfrak{E}aD+\mathfrak{E}aE+\mathfrak{E}aF+\mathfrak{E}aG+\mathfrak{E}aH+\mathfrak{E}aI+\ldots+\mathfrak{E}iI+\text{etc} \\
+\mathfrak{F}aA+\mathfrak{F}aB+\mathfrak{F}aC+\mathfrak{F}aD+\mathfrak{F}aE+\mathfrak{F}aF+\mathfrak{F}aG+\mathfrak{F}aH+\mathfrak{F}aI+\ldots+\mathfrak{F}iI+\text{etc} \\
+\mathfrak{G}aA+\mathfrak{G}aB+\mathfrak{G}aC+\mathfrak{G}aD+\mathfrak{G}aE+\mathfrak{G}aF+\mathfrak{G}aG+\mathfrak{G}aH+\mathfrak{G}aI+\ldots+\mathfrak{G}iI+\text{etc} \\
+\mathfrak{H}aA+\mathfrak{H}aB+\mathfrak{H}aC+\mathfrak{H}aD+\mathfrak{H}aE+\mathfrak{H}aF+\mathfrak{H}aG+\mathfrak{H}aH+\mathfrak{H}aI+\ldots+\mathfrak{H}iI+\text{etc} \\
+\mathfrak{I}aA+\mathfrak{I}aB+\mathfrak{I}aC+\mathfrak{I}aD+\mathfrak{I}aE+\mathfrak{I}aF+\mathfrak{I}aG+\mathfrak{I}aH+\mathfrak{I}aI+\ldots+\mathfrak{I}iI+\text{etc} \\
+\text{etc}+\text{etc}+\text{etc}+\text{etc}+\text{etc}+\text{etc}+\text{etc}+\text{etc}+\text{etc}+\ldots+\text{etc}+\text{etc}
\end{array}\right.$$

$$5\ \text{Sit}\ \left[\begin{array}{l} \overset{1}{A}z^{\mu}+\overset{2}{B}z^{\mu+\delta}+\overset{3}{C}z^{\mu+2\delta}+\overset{4}{D}z^{\mu+3\delta}+\overset{5}{E}z^{\mu+4\delta}+\text{etc}=p \\ az^{\nu}+bz^{\nu+\delta}+cz^{\nu+2\delta}+dz^{\nu+3\delta}+ez^{\nu+4\delta}+\text{etc}=q \\ \mathfrak{A}z^{\pi}+\mathfrak{B}z^{\pi+\delta}+\mathfrak{C}z^{\pi+2\delta}+\mathfrak{D}z^{\pi+3\delta}+\mathfrak{E}z^{\pi+4\delta}+\text{etc}=r \end{array}\right]$$

Erit Producti rqp Terminus $n$tus generalis

$$rqp\urcorner n = {}^{n+2}\overset{rqp}{C}z^{\mu+\nu+\pi+(n-1)\delta}$$

Ergo

$$rqp=[{}^{3}\overset{rqp}{C}+{}^{4}\overset{rqp}{C}z^{\delta}+{}^{5}\overset{rqp}{C}z^{2\delta}+{}^{6}\overset{rqp}{C}z^{3\delta}+{}^{7}\overset{rqp}{C}z^{4\delta}\ldots+{}^{n+2}\overset{rqp}{C}z^{(n-1)\delta}]\,z^{\mu+\nu+\pi}$$

unde, poſito $\mu+\nu+\pi = \text{n}$, fit [Tab. IV. p. LX.]

TABULA I. SERIERUM IN SERIES MULTIPLICATIO.

$$
rqp = \left\{
\begin{array}{l}
\mathfrak{A}aA\,z^{\Pi} \\
+\left[\begin{array}{l}\mathfrak{A}aB + \mathfrak{A}bA \\ \qquad\;\; + \mathfrak{B}aA\end{array}\right\} z^{\Pi+\partial} \\
+\left[\begin{array}{l}\mathfrak{A}aC + \mathfrak{A}bB + \mathfrak{A}cA \\ \qquad\;\; + \mathfrak{B}aB + \mathfrak{B}bA \\ \qquad\qquad\quad + \mathfrak{C}aA\end{array}\right\} z^{\Pi+2\partial} \\
+\left[\begin{array}{l}\mathfrak{A}aD + \mathfrak{A}bC + \mathfrak{A}cB + \mathfrak{A}dA \\ \qquad\;\; + \mathfrak{B}aC + \mathfrak{B}bB + \mathfrak{B}cA \\ \qquad\qquad\quad + \mathfrak{C}aB + \mathfrak{C}bA \\ \qquad\qquad\qquad\quad + \mathfrak{D}aA\end{array}\right\} z^{\Pi+3\partial} \\
+\left[\begin{array}{l}\mathfrak{A}aE + \mathfrak{A}bD + \mathfrak{A}cC + \mathfrak{A}dB + \mathfrak{A}eA \\ \qquad\;\; + \mathfrak{B}aD + \mathfrak{B}bC + \mathfrak{B}cB + \mathfrak{B}dA \\ \qquad\qquad\quad + \mathfrak{C}aC + \mathfrak{C}bB + \mathfrak{C}cA \\ \qquad\qquad\qquad\quad + \mathfrak{D}aB + \mathfrak{D}bA \\ \qquad\qquad\qquad\qquad\quad + \mathfrak{E}aA\end{array}\right\} z^{\Pi+4\partial} \\
+ \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc}
\end{array}\right.
$$

$$
6 \text{ Sit} \left\{
\begin{array}{l}
\overset{1}{A z^{A}} + \overset{2}{B z^{B}} + \overset{3}{C z^{C}} + \overset{4}{D z^{D}} + \overset{5}{E z^{E}} + \overset{6}{F z^{F}} + \text{etc} = p \\
a z^{a} + b z^{b} + c z^{c} + d z^{d} + e z^{e} + f z^{f} + \text{etc} = q \\
\mathfrak{A} z^{\mathfrak{A}} + \mathfrak{B} z^{\mathfrak{B}} + \mathfrak{C} z^{\mathfrak{C}} + \mathfrak{D} z^{\mathfrak{D}} + \mathfrak{E} z^{\mathfrak{E}} + \mathfrak{F} z^{\mathfrak{F}} + \text{etc} = r
\end{array}\right\}.
$$

Erit Producti rqp Terminus $n$tus generalis

$$
rqp\,\mathrm{T}n = \left[{}^{n+2}\overset{rqp}{C}1 + {}^{n+2}\overset{rqp}{C}2 + {}^{n+2}\overset{rqp}{C}3 + {}^{n+2}\overset{rqp}{C}4 \ldots\ldots + {}^{n+2}\overset{rqp}{C}\left(\frac{n.\,n+1}{1.\,2}\right)\right] z^{*}
$$

Ergo

$$
rqp = {}^{3}\overset{rqp}{C}1 .\, z^{*} + \left({}^{4}\overset{rqp}{C}1 + {}^{4}\overset{rqp}{C}2 + {}^{4}\overset{rqp}{C}3\right) z^{*} + \left({}^{5}\overset{rqp}{C}1 + {}^{5}\overset{rqp}{C}2 + {}^{5}\overset{rqp}{C}3 \ldots + {}^{5}\overset{rqp}{C}6\right) z^{*} + \text{etc}.
$$

Et [Tab. IV. p. LX]

$$
rqp = \left\{
\begin{array}{l}
\mathfrak{A}aA\,z^{\mathfrak{A}+a+A} \\
+ \mathfrak{A}aB\,z^{\mathfrak{A}+a+B} + \mathfrak{A}bA\,z^{\mathfrak{A}+b+A} \\
\qquad\qquad\qquad\quad + \mathfrak{B}aA\,z^{\mathfrak{B}+a+A} \\
+ \mathfrak{A}aC\,z^{\mathfrak{A}+a+C} + \mathfrak{A}bB\,z^{\mathfrak{A}+b+B} + \mathfrak{A}cA\,z^{\mathfrak{A}+c+A} \\
\qquad\qquad\qquad\quad + \mathfrak{B}aB\,z^{\mathfrak{B}+a+B} + \mathfrak{B}bA\,z^{\mathfrak{B}+b+A} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad + \mathfrak{C}aA\,z^{\mathfrak{C}+a+A} \\
\quad \text{etc} \qquad\qquad \text{etc} \qquad\qquad \text{etc} \qquad\qquad \text{etc}
\end{array}\right.
$$

TABULA I.

TABULA I. SERIERUM IN SERIES MULTIPLICATIO.

C. Productum a Seriebus quatuor

$$
7\ \text{Sit}\ \left\{\begin{array}{llllll}
\overset{1}{A z^{\mu}} & + \overset{2}{B z^{\mu+\delta}} & + \overset{3}{C z^{\mu+2\delta}} & + \overset{4}{D z^{\mu+3\delta}} & + \overset{5}{E z^{\mu+4\delta}} & + \text{etc} = p \\
a z^{\nu} & + b z^{\nu+\delta} & + c z^{\nu+2\delta} & + d z^{\nu+3\delta} & + e z^{\nu+4\delta} & + \text{etc} = q \\
\mathfrak{A} z^{\pi} & + \mathfrak{B} z^{\pi+\delta} & + \mathfrak{C} z^{\pi+2\delta} & + \mathfrak{D} z^{\pi+3\delta} & + \mathfrak{E} z^{\pi+4\delta} & + \text{etc} = r \\
\mathfrak{a} z^{\sigma} & + \mathfrak{b} z^{\sigma+\delta} & + \mathfrak{c} z^{\sigma+2\delta} & + \mathfrak{d} z^{\sigma+3\delta} & + \mathfrak{e} z^{\sigma+4\delta} & + \text{etc} = s
\end{array}\right\}
$$

Erit Producti $srqp$ Terminus $n$tus generalis

$$srqp\rceil n = {}^{n+3}\overset{srqp}{D} z^{\mu+\nu+\pi+\sigma+(n-1)\delta}$$

Ergo $srqp = [{}^{4}\overset{s..p}{D} + {}^{5}\overset{s..p}{D} z^{\delta} + {}^{6}\overset{s..p}{D} z^{2\delta} + {}^{7}\overset{s..p}{D} z^{3\delta} \ldots + {}^{n+3}\overset{s..p}{D} z^{(n-1)\delta}] z^{\mu+\nu+\pi+\sigma}$

unde, poſito $\mu + \nu + \pi + \sigma = \Sigma$, fit [Tab. IV. p. LX]

$$
srqp = \left\{\begin{array}{l}
\mathfrak{a}\mathfrak{A}aA z^{\Sigma} \\
+\left[\begin{array}{r}
\mathfrak{a}\mathfrak{A}aB + \mathfrak{a}\mathfrak{A}bA \\
+ \mathfrak{a}\mathfrak{B}aA \\
+ \mathfrak{b}\mathfrak{A}aA
\end{array}\right] z^{\Sigma+\delta} \\
+\left[\begin{array}{r}
\mathfrak{a}\mathfrak{A}aC + \mathfrak{a}\mathfrak{A}bB + \mathfrak{a}\mathfrak{A}cA \\
+ \mathfrak{a}\mathfrak{B}aB + \mathfrak{a}\mathfrak{B}bA \\
+ \mathfrak{a}\mathfrak{C}aA \\
+ \mathfrak{b}\mathfrak{A}aB + \mathfrak{b}\mathfrak{A}bA \\
+ \mathfrak{b}\mathfrak{B}aA \\
+ \mathfrak{c}\mathfrak{A}aA
\end{array}\right] z^{\Sigma+2\delta} \\
+\left[\begin{array}{r}
\mathfrak{a}\mathfrak{A}aD + \mathfrak{a}\mathfrak{A}bC + \mathfrak{a}\mathfrak{A}cB + \mathfrak{a}\mathfrak{A}dA \\
+ \mathfrak{a}\mathfrak{B}aC + \mathfrak{a}\mathfrak{B}bB + \mathfrak{a}\mathfrak{B}cA \\
+ \mathfrak{a}\mathfrak{C}aB + \mathfrak{a}\mathfrak{C}bA \\
+ \mathfrak{a}\mathfrak{D}aA \\
+ \mathfrak{b}\mathfrak{A}aC + \mathfrak{b}\mathfrak{A}bB + \mathfrak{b}\mathfrak{A}cA \\
+ \mathfrak{b}\mathfrak{B}aB + \mathfrak{b}\mathfrak{B}bA \\
+ \mathfrak{b}\mathfrak{C}aA \\
+ \mathfrak{c}\mathfrak{A}aB + \mathfrak{c}\mathfrak{A}bA \\
+ \mathfrak{c}\mathfrak{B}aA \\
+ \mathfrak{d}\mathfrak{A}aA
\end{array}\right] z^{\Sigma+3\delta} \\
\text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc} \quad \text{etc}
\end{array}\right.
$$

D. Productum a Seriebus pluribus quotcunque

8. Sint Series diverſae quotcunque, p, q, r, s, t .... numero m,
Erit Producti .... $tsrqp$ Terminus $n$tus generalis

$$\ldots tsrqp\rceil n = {}^{n+m-1}\overset{\ldots rqp}{N} z^{\mu+\nu+\pi+\sigma+\tau\ldots+(n-1)\delta}$$

unde, poſito $\mu+\nu+\pi+\sigma+\tau\ldots\ldots = \Omega$, fit

$$\ldots tsrqp = \left[{}^{m}\overset{\ldots rqp}{N} + {}^{m+1}\overset{\ldots rqp}{N}z^{\delta} + {}^{m+2}\overset{\ldots rqp}{N}z^{2\delta} + {}^{m+3}\overset{\ldots rqp}{N}z^{3\delta}\ldots + {}^{n+m-1}\overset{\ldots rqp}{N}z^{(n-1)\delta}\right]z^{\Omega}$$

et poſito $m = 1,\ 2,\ 3,\ 4\ \ldots$ ſucceſſive

$$\left.\begin{array}{l} \text{fit } \ldots tsrqp = p,\ qp,\ rqp,\ srqp,\ \ldots \\ \text{et } N = A,\ B,\ C,\ D,\ \ldots \\ \text{et } \Omega = \mu,\ \mathrm{N},\ \Pi,\ \Sigma,\ \ldots \end{array}\right\}\text{ reſpective}$$

quod ſi ponitur $p = q = r = s = t = u = etc$, fit ..... $utsrqp = p^m$

E. Summa Productorum a Seriebus pluribus diverſis

9. Sint Series diverſae quotcunque p, q, r, s, t, ...

$$\begin{array}{llllll}
\text{Erit } \alpha p = & \alpha^1\overset{p}{A}z^{\mu} + & \alpha^2\overset{p}{A}z^{\mu+\delta} + & \alpha^3\overset{p}{A}z^{\mu+2\delta} + & \alpha^4\overset{p}{A}z^{\mu+3\delta} + & \alpha^5\overset{p}{A}z^{\mu+4\delta} + \text{etc} \\
\beta qp = & & \beta^2\overset{qp}{B}z^{\mathrm{N}} + & \beta^3\overset{qp}{B}z^{\mathrm{N}+\delta} + & \beta^4\overset{qp}{B}z^{\mathrm{N}+2\delta} + & \beta^5\overset{qp}{B}z^{\mathrm{N}+3\delta} + \text{etc} \\
\gamma rqp = & & & \gamma^3\overset{rqp}{C}z^{\Pi} + & \gamma^4\overset{rqp}{C}z^{\Pi+\delta} + & \gamma^5\overset{rqp}{C}z^{\Pi+2\delta} + \text{etc} \\
\delta srqp = & & & & \delta^4\overset{s..p}{D}z^{\Sigma} + & \delta^5\overset{s..p}{D}z^{\Sigma+\delta} + \text{etc} \\
\varepsilon tsrqp = & & & & & \varepsilon^5\overset{t\ldots p}{E}z^{T} + \text{etc} \\
\text{etc} & & & & & \text{etc}
\end{array}$$

Ergo Productorum a Seriebus pluribus quotcunque Summa

$$\alpha p + \beta qp + \gamma rqp + \delta srqp + \varepsilon tsrqp + \zeta utsrqp + etc =$$

$$= \left\{\begin{array}{llllll}
\alpha^1\overset{p}{A}z^{\mu} & & & & & \\
+\alpha^2 Az^{\mu+\delta} + & \beta^2\overset{qp}{B}z^{\mathrm{N}} & & & & \\
+\alpha^3 Az^{\mu+2\delta} + & \beta^3 Bz^{\mathrm{N}+\delta} & +\gamma^3\overset{rqp}{C}z^{\Pi} & & & \\
+\alpha^4 Az^{\mu+3\delta} + & \beta^4 Bz^{\mathrm{N}+2\delta} & +\gamma^4 Cz^{\Pi+\delta} + & \delta^4\overset{srqp}{D}z^{\Sigma} & & \\
+\alpha^5 Az^{\mu+4\delta} + & \beta^5 Bz^{\mathrm{N}+3\delta} & +\gamma^5 Cz^{\Pi+2\delta} + & \delta^5 Dz^{\Sigma+\delta} + & \varepsilon^5\overset{tsrqp}{E}z^{T} & \\
\text{etc} & \text{etc} & \text{etc} & \text{etc} & \text{etc} & \text{etc}
\end{array}\right.$$

TABULA II

SERIERUM PER SERIES DIVISIO

Hac Tabula continentur

Canones generales Quotientium

$$\frac{1}{1-\alpha-\beta-\gamma-\delta-\varepsilon-\text{etc}}$$ Propoſ. 1.

$$\frac{1}{\alpha+\beta+\gamma+\delta+\varepsilon+\text{etc}}$$ Propoſ. 2.

$$\frac{a z^{\mu}+b z^{\mu+\Delta}+c z^{\mu+2\Delta}+\text{etc}}{\alpha z^{\nu}+\beta z^{\nu+\Delta}+\gamma z^{\nu+2\Delta}+\text{etc}}$$ Propoſ. 3.

$$\frac{a z^{a}+b z^{b}+c z^{c}+d z^{d}+\text{etc}}{\alpha z^{\alpha}+\beta z^{\beta}+\gamma z^{\gamma}+\delta z^{\delta}+\text{etc}}$$ Propoſ. 4.

$$\frac{a+b+c+d+e+\text{etc}}{\alpha+\beta+\gamma+\delta+\varepsilon+\text{etc}}$$ Propoſ. 5.

$$\frac{1}{\alpha z^{\nu}+\beta z^{\nu+\Delta}+\gamma z^{\nu+2\Delta}+\text{etc}}$$ Propoſ. 6.

$$\frac{a+b+c+d+e+\text{etc}}{\alpha z^{\nu}+\beta z^{\nu+\Delta}+\gamma z^{\nu+2\Delta}+\text{etc}}$$ Propoſ. 7.

Numeri Complexionum ſinguli quorſum pertineant, patet ex adiecto ſingulis Propoſitionibus Indiculo

$$\begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6 \ldots \\ \alpha, & \beta, & \gamma, & \delta, & \varepsilon, & \zeta \ldots\end{pmatrix} \text{ vel } \begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6 \ldots \\ \beta, & \gamma, & \delta, & \varepsilon, & \zeta, & \eta \ldots\end{pmatrix}$$

TABULA II.

SERIERUM PER SERIES DIVISIO.

1. Sit $\frac{1}{1-\alpha-\beta-\gamma-\delta-\varepsilon-\text{etc}}=\frac{1}{1-q}$

Erit
$$\frac{1}{1-q}\left\{\begin{aligned}&=1+q+q^{2}+q^{3}+q^{4}+q^{5}+q^{6}+\text{etc}\\&=1+a'A+b'B+c'C+d'D+e'E+f'F+\text{etc}\qquad(P)\\&=1+a^{1}A+(a^{2}A+b^{2}B)+(a^{3}A+b^{3}B+c^{3}C)+\text{etc}\qquad(Q)\end{aligned}\right.$$

$$\begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots\ldots\\ \alpha, & \beta, & \gamma, & \delta, & \varepsilon, & \zeta, & \eta, & \ldots\ldots\end{pmatrix}$$

Ergo

$$\frac{1}{1-q}=\left\{\begin{aligned}&1\\&+\alpha+\beta+\gamma+\text{etc}\\&+\alpha^{2}+2\alpha\beta+2\alpha\gamma\ldots+\beta^{2}+2\beta\gamma\ldots+\gamma^{2}\ldots+\text{etc}\\&+\alpha^{3}+3\alpha^{2}\beta+3\alpha^{2}\gamma\ldots+3\alpha\beta^{2}+6\alpha\beta\gamma\ldots+3\alpha\gamma^{2}\ldots\\&\qquad+\beta^{3}+3\beta^{2}\gamma\ldots+3\beta\gamma^{2}\ldots+\gamma^{3}\ldots+\text{etc}\\&+\text{etc}\qquad\text{etc}\qquad\text{etc}\qquad\text{etc}\qquad(P)\end{aligned}\right.$$

sive $\frac{1}{1-q}=1+\alpha+(\beta+\alpha^{2})+(\gamma+2\alpha\beta+\alpha^{3})+\begin{pmatrix}\delta+[2\alpha\gamma+\beta^{2}]\\+3\alpha^{2}\beta+\alpha^{4}\end{pmatrix}+\text{etc}\,(Q)$

2. Sit $\frac{1}{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta+\text{etc}}=\frac{1}{q}$

Erit
$$\frac{1}{q}\left\{\begin{aligned}&=\frac{1}{\alpha}-\frac{a'A}{\alpha^{2}}+\frac{b'B}{\alpha^{3}}-\frac{c'C}{\alpha^{4}}+\frac{d'D}{\alpha^{5}}-\frac{e'E}{\alpha^{6}}+\frac{f'F}{\alpha^{7}}-\text{etc}\qquad(R)\\&=\frac{1}{\alpha}-\frac{a^{1}A}{\alpha^{2}}-\left(\frac{a^{2}A}{\alpha^{2}}-\frac{b^{2}B}{\alpha^{3}}\right)-\left(\frac{a^{3}A}{\alpha^{2}}-\frac{b^{3}B}{\alpha^{3}}+\frac{c^{3}C}{\alpha^{4}}\right)-\text{etc}\qquad(S)\end{aligned}\right.$$

$$\begin{pmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7 & \ldots\ldots\\ \beta, & \gamma, & \delta, & \varepsilon, & \zeta, & \eta, & \theta & \ldots\ldots\end{pmatrix}$$

Ergo

$$\frac{1}{q}=\left\{\begin{aligned}&\frac{1}{\alpha}\\&-\frac{\beta+\gamma+\delta+\text{etc}}{\alpha^{2}}\\&+\frac{\beta^{2}+2\beta\gamma+2\beta\delta\ldots+\gamma^{2}+2\gamma\delta\ldots+\gamma^{3}\ldots+\text{etc}}{\alpha^{3}}\\&-\text{etc}\qquad\text{etc}\qquad\text{etc}\qquad\text{etc}\qquad\text{etc}\qquad(R)\end{aligned}\right.$$

Sive $\frac{1}{q}=\frac{1}{\alpha}-\frac{\beta}{\alpha^{2}}-\left(\frac{\gamma}{\alpha^{2}}-\frac{\beta^{2}}{\alpha^{3}}\right)-\left(\frac{\delta}{\alpha^{2}}-\frac{2\beta\gamma}{\alpha^{3}}+\frac{\beta^{3}}{\alpha^{4}}\right)-\text{etc}\,(S)$

3. Sit $\dfrac{a z^{\mu} + b z^{\mu+\Delta} + c z^{\mu+2\Delta} + d z^{\mu+3\Delta} \ldots + n z^{\mu+(n-1)\Delta}}{\alpha z^{\nu} + \beta z^{\nu+\Delta} + \gamma z^{\nu+2\Delta} + \delta z^{\nu+3\Delta} + \varepsilon z^{\nu+4\Delta} + \text{etc}} = \dfrac{p}{q}$

Erit Quotientis $\dfrac{p}{q}$ Terminus $n$tus generalis

$$\frac{p}{q}\Big]n = \left[\begin{aligned} &\frac{\overset{0}{n}}{\alpha^{1}} - \frac{\overset{-1}{n} a^{1} A + \overset{-2}{n} a^{2} A \ldots + b a^{n-2} A + a a^{n-1} A}{\alpha^{2}} \\ &+ \frac{\overset{-2}{n} b^{2} B + \overset{-3}{n} b^{3} B \ldots + b b^{n-2} B + a b^{n-1} B}{\alpha^{3}} \\ &- \frac{\overset{-3}{n} c^{3} C + \overset{-4}{n} c^{4} C \ldots + b c^{n-3} C + a c^{n-1} C}{\alpha^{4}} \\ &+ \text{etc} \qquad \text{etc} \qquad \text{etc} \end{aligned}\right] z^{\mu-\nu+(n-1)\Delta}$$

Ergo

$$\frac{p}{q} = \left\{\begin{aligned} &\frac{a}{\alpha} z^{\mu-\nu} \\ &+ \left[\frac{b}{\alpha} - \frac{a a^{1} A}{\alpha^{2}}\right] z^{\mu-\nu+\Delta} \\ &+ \left[\frac{c}{\alpha} - \frac{b a^{1} A + a a^{2} A}{\alpha^{2}} + \frac{a b^{2} B}{\alpha^{3}}\right] z^{\mu-\nu+2\Delta} \\ &+ \left[\frac{d}{\alpha} - \frac{c a^{1} A + b a^{2} A + a a^{3} A}{\alpha^{2}} + \frac{b b^{2} B + a b^{3} B}{\alpha^{3}} - \frac{a c^{3} C}{\alpha^{4}}\right] z^{\mu-\nu+3\Delta} \\ &+ \left[\frac{e}{\alpha} - \frac{d a^{1} A + c a^{2} A + b a^{3} A + a a^{4} A}{\alpha^{2}} + \frac{c b^{2} B + b b^{3} B + a b^{4} B}{\alpha^{3}} - \frac{b c^{3} C + a c^{4} C}{\alpha^{4}} + \frac{a b^{4} D}{\alpha^{5}}\right] z^{\mu-\nu+4\Delta} \\ &\text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \end{aligned}\right.$$

$$(1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7 \ldots\ldots)$$

unde, poſito $\mu - \nu = M$, fit [Tab. III. p. LIX.] poſitis ibi $\left(\begin{matrix}\text{pro} & \alpha, & \beta, & \gamma, & \delta, & \varepsilon \ldots\ldots \\ \text{hic} & \beta, & \gamma, & \delta, & \varepsilon, & \zeta \ldots\ldots\end{matrix}\right)$

$$\frac{p}{q} = \left\{\begin{array}{l} \frac{a}{\alpha} z^{M} \\ + \left[\frac{b}{\alpha} - \frac{a\beta}{\alpha^2}\right] z^{M+\Delta} \\ + \left[\frac{c}{\alpha} - \frac{b\beta + a\gamma}{\alpha^2} + \frac{a\beta^2}{\alpha^3}\right] z^{M+2\Delta} \\ + \left[\frac{d}{\alpha} - \frac{c\beta + b\gamma + a\delta}{\alpha^2} + \frac{b\beta^2 + a.2\beta\gamma}{\alpha^3} - \frac{a\beta^3}{\alpha^4}\right] z^{M+3\Delta} \\ + \left[\frac{e}{\alpha} - \frac{d\beta + c\gamma + b\delta + a\varepsilon}{\alpha^2} + \frac{c\beta^2 + b.2\beta\gamma + a(2\beta\delta + \gamma^2)}{\alpha^3} - \frac{b\beta^3 + a.3\beta^2\gamma}{\alpha^4} + \frac{a\beta^4}{\alpha^5}\right] z^{M+4\Delta} \\ \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \end{array}\right.$$

4. Sit $\dfrac{a z^{a} + b z^{b} + c z^{c} + d z^{d} + e z^{e} + \text{etc}}{\alpha z^{\alpha} + \beta z^{\beta} + \gamma z^{\gamma} + \delta z^{\delta} + \varepsilon z^{\varepsilon} + \text{etc}} = \dfrac{p}{q}$

$$\text{Erit } \frac{p}{q} = \left\{\begin{array}{l} \frac{a}{\alpha} z^{*} \\ + \left[\frac{b}{\alpha} - \frac{a\mathfrak{a}^{1}A1}{\alpha^2}\right] z^{*} \\ + \left[\frac{c}{\alpha} - \frac{b\mathfrak{a}^{1}A1 + a\mathfrak{a}^{2}A1}{\alpha^2} + \frac{a\mathfrak{b}^{2}B1}{\alpha^3}\right] z^{*} \\ \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \end{array}\right.$$

$$\frac{p}{q} = \left\{\begin{array}{l} \frac{a}{\alpha} z^{a-\alpha} \\ + \left[\frac{b}{\alpha} z^{b-\alpha} - \frac{a\beta}{\alpha^2} z^{a+\beta-2\alpha} \right. \\ + \left[\frac{c}{\alpha} z^{c-\alpha} - \frac{b\beta z^{b+\beta-2\alpha} + a\gamma z^{a+\gamma-2\alpha}}{\alpha^2} + \frac{a\beta^2}{\alpha^3} z^{a+2\beta-3\alpha} \right. \\ \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \qquad \text{etc} \end{array}\right. \qquad \left(\begin{matrix}1, & 2, & 3, & 4, & 5, \ldots\ldots \\ \beta, & \gamma, & \delta, & \varepsilon, & \zeta, \ldots\ldots\end{matrix}\right)$$

5. Sit $\dfrac{a + b + c + d + e + \text{etc}}{\alpha + \beta + \gamma + \delta + \varepsilon + \text{etc}} = \dfrac{p}{q}$

$$\text{Erit } \frac{p}{q} \left\{\begin{array}{l} = p.\frac{1}{q} = pR \;=\; pS \quad [2] \\ = \text{Coefficientibus Seriei } \frac{p}{q} \; [3] \end{array}\right.$$